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Self-Similar Behaviour for the Equation of Fast Nonlinear Diffusion

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Self-similar behaviour for the equation of fast nonlinear diffusion

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We consider the Cauchy problem for the equation $u_t = \nabla\cdot(u^{-n}\nabla u)$ in $\mathbb{R}^N$ for the cases $N>2$ with $2/N \leqslant n < 1$ and $N=2$ with $n=1$ for which the time-asymptotic behaviour of finite mass solutions has not previously been established. For $N>2$ with $n=2/N$ the behaviour as $t\to+\infty$ is shown to take an unusual self-similar form. For $N>2$ with $2/N<n<1$ and $N=2$ with $n=1$ solutions extinguish in finite time. In the former case we show that the behaviour close to the extinction time is given by a similarity solution of the second kind and we derive a number of results for the similarity exponent. For $N=2$ with $n=1$ the solution to the Cauchy problem is not uniquely specified, and we characterize the possible types of solution and establish their behaviour close to extinction. We also indicate how physical considerations can lead to a unique selection from among the available solutions. The limit $N\to\infty$ is also analysed, illustrating how the various types of asymptotic behaviour arise from the evolution over earlier times.

1. Introduction

This paper is concerned with the Cauchy problem for the N-dimensional equation of 'fast' nonlinear diffusion, namely

$$\partial u/\partial t = \nabla\cdot(u^{-n}\nabla u) \tag{1.1}$$

with $n>0, N\geqslant 2$. The equation (1.1) arises in a wide range of applications of which we note the following.

1. $n=1$ is applicable to the spreading of microscopic droplets (Lopez *et al.* 1976; de Gennes 1984). More general values of n have also been considered in this context (see, for example, Starov 1983).

2. $n=1$ also arises in plasma applications (Lonngren & Hirose 1976; Berryman & Holland 1982). Values in the range $0<n<1$ are also of interest (see, for example, Berryman & Holland 1978).

3. The range $0<n<1$ occurs in models of the diffusion of impurities in silicon (King 1988). For illustrative purposes we give here a derivation of a simplified form of the relevant model. It is assumed that the impurity can exist in two states, namely mobile individual atoms (concentration c) and immobile clusters (concentration c_c), each of the latter being made up of m atoms. The clustering reaction is then

$$m\mathrm{A} \rightleftharpoons \mathrm{A}_m,$$

where A denotes an impurity atom and A_m a cluster. The simplest model describing these processes assumes that this reaction is in equilibrium, so that

$$c_c = Kc^m, \tag{1.2}$$

where K is a constant, and that the redistribution of the mobile atoms can be described by a linear diffusion term with constant diffusivity D, so that

$$\frac{\partial}{\partial t}(c+c_c) = D\nabla^2 c. \tag{1.3}$$

Writing $u = c+c_c$ and using (1.2), it is clear that, under appropriate scalings, the high concentration behaviour for which $c_c \gg c$ can be described by (1.1) with

$$n = (m-1)/m. \tag{1.4}$$

As time increases, however, the maximum concentration will decrease, so the condition $c_c \gg c$ eventually ceases to be valid. On sufficiently long timescales we have $c_c \ll c$ and the problem becomes dominated by linear diffusion. The results of this paper thus provide descriptions of the intermediate asymptotic behaviour of solutions to (1.2) and (1.3).

In addition to its physical relevance, (1.1) has been the subject of a great deal of mathematical analysis. Some indication of this is provided by the review article of Aronson (1986) which, however, is largely concerned with the 'slow' diffusion case ($n < 0$) rather than the 'fast' diffusion case ($n > 0$) to which the current paper is devoted.

We are concerned here with the Cauchy problem in which (1.1) is subject to the initial condition

$$\text{at} \quad t = 0, \quad u = I(\boldsymbol{x}) \quad \text{for} \quad \boldsymbol{x} \in \mathbb{R}^N, \tag{1.5}$$

where the total mass

$$M^* = \int_{\mathbb{R}^N} I(\boldsymbol{x})\,\mathrm{d}V$$

is finite, and we now indicate some of the known results for (1.1) subject to (1.5) which are of relevance here.

(*a*) For $n < \min(1, 2/N)$, it is well known (see, for example, Friedman & Kamin 1980) that the large-time behaviour is given by the appropriate instantaneous source similarity solution which satisfies

$$\text{at} \quad t = 0, \quad u = M^*\delta(\boldsymbol{x}) \quad \text{for} \quad \boldsymbol{x} \in \mathbb{R}^N,$$

and takes the form (Landau & Lifschitz 1959)

$$u = t^{-N/(2-nN)} f(|\boldsymbol{x}|/t^{1/(2-nN)}), \tag{1.6}$$

where, for $n > 0$,

$$f(\eta) = \left(\frac{n}{2(2-nN)}(a^2+\eta^2)\right)^{-1/n}. \tag{1.7}$$

The constant a can be determined using conservation of mass, the relation

$$\int_{\mathbb{R}^N} u(\boldsymbol{x}, t)\,\mathrm{d}V = M^* \tag{1.8}$$

being valid for all t in this parameter range.

The role of such similarity solutions in describing the large-time behaviour has been known at least since the work of Zel'dovich & Barenblatt (1958). The time-asymptotic behaviour for $n \geqslant 2/N$ (so that the solution (1.6) and (1.7) is not applicable) has not previously been determined, however, and this is the main goal of the current paper.

(*b*) For $N = 1$ with $1 \leqslant n < 2$ it is known (Esteban *et al.* 1988) that the problem (1.1) and (1.5) is not uniquely specified. The maximal solution conserves mass and its large-time behaviour is again given by (1.6) and (1.7), but other solutions also exist which lose mass to infinity.

(*c*) For $N > 2$ with $2/N < n < 1$ finite mass solutions do not conserve mass and they vanish in finite time (Benilan & Crandall 1981).

(*d*) For $N = 1$ with $n \geqslant 2$, $N = 2$ with $n > 1$ and $N > 2$ with $n \geqslant 1$ the Cauchy problem for equation (1.1) has no finite mass solutions (Herrero 1989; Vazquez 1992), and these parameter ranges are therefore not considered here.

For the remainder of this paper we shall largely restrict attention to the radially symmetric version of (1.1) and (1.5), so we consider

$$\frac{\partial u}{\partial t} = \frac{1}{r^{N-1}} \frac{\partial}{\partial r}\left(r^{N-1} u^{-n} \frac{\partial u}{\partial r}\right), \tag{1.9}$$

subject to

$$\left.\begin{array}{lll} \text{at} & t = 0, & u = I(r), \\ \text{at} & r = 0, & r^{N-1} u^{-n}\, \partial u/\partial r = 0, \\ \text{as} & r \to \infty, & u \to 0, \end{array}\right\} \tag{1.10}$$

and we write

$$M = \int_0^\infty r^{N-1} I(r)\, \mathrm{d}r, \tag{1.11}$$

which is be taken to be finite. We also assume that $I(r) \geqslant 0$ holds for all r.

In this paper we consider the following issues which have not previously been settled.

(I) For $N > 2$ with $n = 2/N$ solutions preserve mass, but their large-time behaviour is evidently not given by (1.6). In §2 we determine the large-time behaviour, showing that it is asymptotically self-similar and of an unusual form.

(II) For $N > 2$ with $2/N < n < 1$ the asymptotic behaviour of solutions close to their time of extinction ($t = t_c$) has not been established. In §4 we show that the relevant behaviour is given by a similarity solution of the second kind. Thus the exponent β in the relevant similarity solution

$$u \sim (t_c - t)^{(1-2\beta)/n} f(r/(t_c - t)^{\beta}) \quad \text{as} \quad t \to t_c^-$$

cannot be calculated from an integral constraint, the mass not being conserved, but is determined by a nonlinear eigenvalue problem.

(III) For $N = 2$ with $n = 1$ it has been shown (Vazquez 1992) that finite mass solutions which vanish in finite time exist, but little else is known. In §5 we show that the problem (1.9) and (1.10) is again not uniquely specified for these parameter values but that, in contrast to the case $N = 1$ with $1 \leqslant n < 2$, the maximal solution loses mass and vanishes in finite time. We also determine the asymptotic behaviour close to the time of extinction.

(IV) By considering the radially symmetric case (1.9) we may permit N to take non-integer values (some motivation for this will be given later). In the range $N > 0$ the problem (1.9) and (1.10) is not uniquely specified when $N < 2$ with $1 \leqslant n < 2/N$ and when $N = 2$ with $n = 1$ (the special cases $N = 1$ and $N = 2$ have already been noted). In §6 we indicate how, in the appropriate limit, the incorporation of a physically motivated regularizing effect selects uniquely from among the available solutions. Section 6 also contains some discussion of our results.

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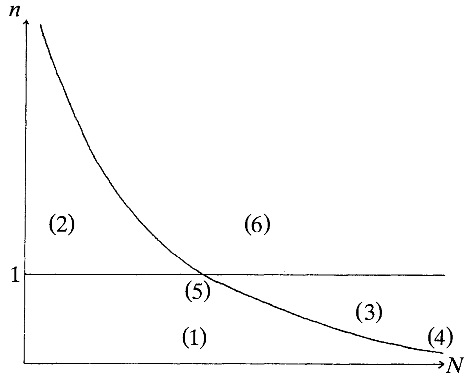

Figure 1. Schematic of parameter ranges. (1) $n < \min(1, 2/N)$, (2) $1 \leqslant n < 2/N$, (3) $2/N < n < 1$, (4) $n = 2/N$ with $n > 2$, (5) $n = 1$ with $N = 2$, (6) $n \geqslant \max(1, 2/N)$ with $N \neq 2$, $n > 1$ with $N = 2$.

Section 3 is rather different from the remainder of the paper since it addresses the limit $N \to \infty$ rather than the time-asymptotic behaviour. This limit is instructive because it enables the evolution over all time to be evaluated for all relevant n and thus indicates how the various different cases arise. This limit is that in which the geometrical aspects of the problem are emphasized.

Finally, it turns out that the behaviour as $r \to \infty$ plays an important role in determining the nature of the solutions, and the relevant results for the possible far-field behaviours are summarized in the Appendix.

The time-asymptotic results of this paper describe timescales on which the form of the solution to (1.1) is independent of the details of the initial conditions. For models such as (1.2) and (1.3), which are well approximated by (1.1) only over a certain concentration range, these results describe the possible forms of the intermediate asymptotic behaviour. Since such timescales are those on which the solution is most sensitive to the value of n, such results are of potential practical use in determining n from experimental observations. It is worth stressing that the qualitative behaviour of solutions to (1.1) can depend strongly on the dimension N as well as on the exponent n.

The various parameter ranges noted above are shown in figure 1. The cases discussed in this paper are (4) (§2), (3) (§4), (5) (§5) and (2) (§6).

In this paper we emphasize the important role played by similarity solutions in each of the various cases. The term solution is used loosely here since these similarity 'solutions' often do not exactly satisfy (1.1), but they do provide self-similar descriptions of the relevant asymptotic behaviour. In several cases this is of a very unusual type, as illustrated by (2.33) and (5.33) in particular.

2. $N > 2$ with $n = 2/N$

In this section we discuss the equation

$$\frac{\partial u}{\partial t} = \frac{1}{r^{N-1}} \frac{\partial}{\partial r}\left(r^{N-1} u^{-2/N} \frac{\partial u}{\partial r}\right). \tag{2.1}$$

As already mentioned, in this borderline case equation (1.1) admits finite mass solutions which conserve mass. An important difference from the case $n < 2/N$ lies in the large-time behaviour, it being evident from (1.6) and (1.7) that the usual similarity solution is not appropriate when $n = 2/N$.

We start by noting the far-field behaviour. When the initial profile $I(r)$ decays much more rapidly than $r^{-2/n}$ as $r \to \infty$ then the far-field behaviour of (1.9) with $n < 2/N$ is described by a separable solution:

$$u \sim \left(\frac{nr^2}{2(2-nN)\,t}\right)^{-1/n} \quad \text{as} \quad r \to \infty;$$

the similarity solution (1.7) evidently has this behaviour. Seeking a comparable expression for (2.1) we assume

$$u \sim t^{N/2} F(r) \quad \text{as} \quad r \to \infty,$$

where $F(r)$ satisfies

$$\tfrac{1}{2}N r^{N-1} F = \frac{\mathrm{d}}{\mathrm{d}r}\left(r^{N-1} F^{-2/N} \frac{\mathrm{d}F}{\mathrm{d}r}\right). \tag{2.2}$$

It is readily seen that a possible balance in (2.2) as $r \to \infty$ is given for $N > 2$ by

$$F \sim ((N-2)^{-1} r^2 \ln r)^{-N/2}. \tag{2.3}$$

The total mass

$$M = \int_0^\infty r^{N-1} u(r,t)\,\mathrm{d}r \tag{2.4}$$

associated with such far-field behaviour is bounded; the corresponding flux

$$-r^{N-1} u^{-2/N} \frac{\partial u}{\partial r} \sim N\left(\frac{\ln r}{(N-2)\,t}\right)^{-(N-2)/2} \quad \text{as} \quad r \to \infty$$

decays only logarithmically for large r.

To determine the large-time behaviour of (2.1) it turns out to be necessary to consider the behaviour as $r \to \infty$ in more detail. This depends rather sensitively on the form of the initial conditions. We concentrate on the case in which the initial data has compact support with

$$I(r) = 0 \quad \text{for} \quad r \geqslant r_0$$

and for definiteness we assume that

$$I(r) \sim A(r_0 - r)^b \quad \text{as} \quad r \to r_0^-$$

for positive constants A and b. As $t \to 0^+$ the solution to (2.1) then satisfies

$$u \sim I(r) \qquad r < r_0, \tag{2.5}$$

$$u \sim t^{Nb/2(N+b)} \phi(\omega) \qquad r = r_0 + O(t^{N/2(N+b)}), \tag{2.6}$$

where $\omega = (r - r_0)\,t^{-N/2(N+b)}$ and $\phi(\omega)$ satisfies

$$\left.\begin{aligned} &\frac{N}{2(N+b)}\left(b\phi - \omega\frac{\mathrm{d}\phi}{\mathrm{d}\omega}\right) = \frac{\mathrm{d}}{\mathrm{d}\omega}\left(\phi^{-2/N}\frac{\mathrm{d}\phi}{\mathrm{d}\omega}\right), \\ &\text{as} \quad \omega \to -\infty, \quad \phi \sim A(-\omega)^b, \\ &\text{as} \quad \omega \to +\infty, \quad \phi \to 0; \end{aligned}\right\} \tag{2.7}$$

for $t \ll 1$ the dominant behaviour close to $r = r_0$ is thus one dimensional.

The solution to (2.7) satisfies

$$\phi \sim (\omega^2/2\,(N-1))^{-N/2} \quad \text{as} \quad \omega \to +\infty,$$

for any value of b, and the important consequence for our purposes is that for $r > r_0$, $t \ll 1$ we have

$$u \sim ((r-r_0)^2/2(N-1)\,t)^{-N/2} \quad \text{for} \quad t^{N/2(N+b)} \ll r-r_0 \ll 1, \tag{2.8}$$

this being independent of the behaviour of $I(r)$ close to $r = r_0$. It follows that for $t \ll 1$ the behaviour is given by (2.5) and (2.6) together with

$$u \sim t^{N/2} F(r) \quad r > r_0, \tag{2.9}$$

where F satisfies (2.2) with

$$\left.\begin{aligned} &\text{as} \quad r \to r_0^+, \quad F \sim ((r-r_0)^2/2(N-1))^{-N/2}, \\ &\text{as} \quad r \to +\infty, \quad F \to 0; \end{aligned}\right\} \tag{2.10}$$

the first of these conditions follows from (2.8). Analysing the behaviour of (2.2) in the limit $r \to \infty$ (for example by linearizing about (2.3)) it may be shown that

$$F^{-2/N} = \frac{r^2}{(N-2)}\left(\ln r - \frac{N}{2(N-2)} \ln\ln r - x_0 + o(1)\right) \quad \text{as} \quad r \to \infty, \tag{2.11}$$

where x_0 is a constant which may in principle be determined by solving (2.2) subject to (2.10). Because (2.2) is invariant under the rescaling $r \to r_0 r, F \to r_0^{-N} F$, it is easily shown that

$$x_0 = \ln r_0 + \gamma_N, \tag{2.12}$$

where the constant γ_N is independent of r_0.

Motivated by (2.11) we introduce the change of variables

$$u = r^{-N} c, \quad x = \ln r,$$

which turns out to provide a convenient formulation for the problem, and we then obtain the convection–diffusion equation

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(c^{-2/N}\left(\frac{\partial c}{\partial x} - Nc\right)\right). \tag{2.13}$$

Writing

$$u(0,t) = U(t),$$

equation (2.13) is subject to

$$\left.\begin{aligned} &\text{as} \quad x \to -\infty, \quad c \sim U(t)\, \mathrm{e}^{Nx}, \\ &\text{as} \quad x \to +\infty, \quad c \sim (x/(N-2)\,t)^{-N/2}, \end{aligned}\right\} \tag{2.14}$$

where $U(t)$ must be determined as part of solution, and where we have assumed that (2.3) describes the far-field behaviour of u. It follows from (2.4) that

$$\int_{-\infty}^{\infty} c(x,t)\, \mathrm{d}x = M. \tag{2.15}$$

As already indicated, to determine the large-time behaviour completely we require a more detailed description of the behaviour as $x \to +\infty$ than that given in (2.14). It follows from (2.11) that

$$c^{-2/N} \sim (x - (N/2(N-2)) \ln x - x_0)/(N-2)\,t \quad \text{for} \quad t \ll 1, \quad x \to +\infty. \tag{2.16}$$

The important feature of the expression (2.16) is that it in fact holds as $x \to +\infty$ for all t. To establish this we note that if we write

$$c - t^{N/2} G(x) \sim C(x,t) \quad \text{as} \quad x \to +\infty,$$

where $\quad G(x) = \mathrm{e}^{Nx} F(\mathrm{e}^x) \quad$ so that $\quad G(x) \sim (x/(N-2))^{-N/2} \quad$ as $\quad x \to +\infty,$

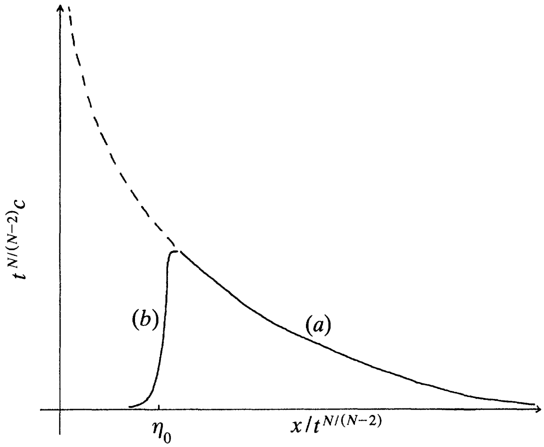

Figure 2. Schematic of behaviour of solution to (2.13) as $t \to +\infty$. (a) g_0, see (2.21). (b) h_0, see (2.28).

then at leading order C is given from (2.13) by

$$\frac{\partial C}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{xC}{t}\right),$$

the general solution to which takes the form

$$C = t^{-1} P(x/t), \tag{2.17}$$

where P is an arbitrary function. Because (2.9) holds, we require that C go to zero more rapidly than $t^{N/2}$ as $t \to 0$ for fixed $x > \ln r_0$, which implies that $P(\sigma)$ must decay faster than $\sigma^{-(N+2)/2}$ as $\sigma \to +\infty$. It thus follows that for all t we have

$$c^{-2/N} = (x - (N/2(N-2)) \ln x - x_0 + o(1))/(N-2)\,t \quad \text{as} \quad x \to +\infty. \tag{2.18}$$

We now determine the large-time behaviour of (2.13) subject to (2.14), (2.15) and (2.18). An asymptotic form consistent with these conditions is that

$$c \sim t^{-N/(N-2)} g_0(\eta) + t^{-2N/(N-2)} \ln t\, g_1(\eta) + t^{-2N/(N-2)} g_2(\eta) \quad \text{as} \quad t \to +\infty, \tag{2.19}$$

where $\eta = x/t^{N/(N-2)}$. It follows that g_0 is given by

$$-\frac{N}{(N-2)}\left(g_0 + \eta \frac{\mathrm{d}g_0}{\mathrm{d}\eta}\right) = -N \frac{\mathrm{d}}{\mathrm{d}\eta}(g_0^{1-2/N}), \tag{2.20}$$

the convection term being the only term on the right-hand side of (2.13) which features at leading order. The required solution of (2.20) is

$$g_0(\eta) = (\eta/(N-2))^{-N/2}. \tag{2.21}$$

The expression (2.21) is evidently not valid for all η and an inner region is needed; this will be discussed shortly. The asymptotic structure is shown schematically in figure 2. The representation (2.19) therefore gives an outer expansion valid for

$$x = O(t^{N/(N-2)}) \quad \text{as} \quad t \to +\infty \quad \text{with} \quad x/t^{N/(N-2)} > \eta_0,$$

where the constant η_0 can be determined from (2.15); setting

$$\int_{\eta_0}^{\infty} g_0(\eta)\,\mathrm{d}\eta = M$$

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yields $$\eta_0 = (N-2)(M/2)^{-2/(N-2)}. \tag{2.22}$$

We shall also require the correction terms g_1 and g_2 in (2.19). We have

$$-\frac{N}{(N-2)}\left(2g_1+\eta\frac{\mathrm{d}g_1}{\mathrm{d}\eta}\right) = -(N-2)\frac{\mathrm{d}}{\mathrm{d}\eta}(g_0^{-2/N}g_1)$$

and

$$g_1-\frac{N}{(N-2)}\left(2g_2+\eta\frac{\mathrm{d}g_2}{\mathrm{d}\eta}\right) = \frac{\mathrm{d}}{\mathrm{d}\eta}\left(g_0^{-2/N}\frac{\mathrm{d}g_0}{\mathrm{d}\eta}\right)-(N-2)\frac{\mathrm{d}}{\mathrm{d}\eta}(g_0^{-2/N}g_2),$$

so that

$$g_1 = (\eta/(N-2))^{-(N+2)/2}A_1, \tag{2.23}$$

$$g_2 = (\eta/(N-2))^{-(N+2)/2}(A_2+(\tfrac{1}{2}(N-2)A_1-\tfrac{1}{8}N^2/(N-2))\ln\eta), \tag{2.24}$$

where A_1 and A_2 are constants of integration. For consistency with (2.18) we then need

$$A_1 = N^3/4(N-2)^3, \quad A_2 = Nx_0/2(N-2),$$

and to this order the outer expansion (2.19) simply reproduces the far-field behaviour given by (2.18).

To complete the analysis of (2.13) in the limit $t\to+\infty$ we must consider an interior layer with scaling

$$x = \eta_0 t^{N/(N-2)}+\frac{N^2}{2(N-2)^2}\ln t+z,$$

with $z = O(1)$. The inner expansion then takes the form

$$c \sim t^{-N/(N-2)}h_0(z)+t^{-2N/(N-2)}h_1(z) \quad \text{as} \quad t\to+\infty \tag{2.25}$$

and the outer solutions provide matching conditions

$$h_0 \sim (\eta_0/(N-2))^{-N/2} \quad \text{as} \quad z\to+\infty, \tag{2.26}$$

$$h_1 \sim -N(\eta_0/(N-2))^{-N/2}(z-(N/2(N-2))\ln\eta_0-x_0)/2\eta_0 \quad \text{as} \quad z\to+\infty. \tag{2.27}$$

In addition we require that

$$h_0, h_1\to 0 \quad \text{as} \quad z\to-\infty.$$

We then have

$$-\frac{N}{N-2}\eta_0\frac{\mathrm{d}h_0}{\mathrm{d}z} = \frac{\mathrm{d}}{\mathrm{d}z}\left(h_0^{-2/N}\left(\frac{\mathrm{d}h_0}{\mathrm{d}z}-Nh_0\right)\right)$$

with solution $$h_0(z) = \left(\frac{\eta_0}{N-2}(1+\mathrm{e}^{-2(z-z_0)})\right)^{-N/2}, \tag{2.28}$$

where z_0 is a constant of integration.

At next order we find that

$$-\frac{N}{(N-2)}h_0-\frac{N^2}{2(N-2)^2}\frac{\mathrm{d}h_0}{\mathrm{d}z}-\frac{N}{(N-2)}\eta_0\frac{\mathrm{d}h_1}{\mathrm{d}z} = \frac{\mathrm{d}^2}{\mathrm{d}z^2}(h_0^{-2/N}h_1)-(N-2)\frac{\mathrm{d}}{\mathrm{d}z}(h_0^{-2/N}h_1),$$

so that

$$-\frac{N}{(N-2)}\int_{-\infty}^{z}h_0(z')\,\mathrm{d}z'-\frac{N^2}{2(N-2)^2}h_0-\frac{N}{N-2}\eta_0 h_1 = \frac{\mathrm{d}}{\mathrm{d}z}(h_0^{-2/N}h_1)-(N-2)h_0^{-2/N}h_1,$$

from which it is easily shown that

$$h_1 \sim -N(\eta_0/(N-2))^{-N/2}(z+1/(N-2)+\kappa_N-z_0)/2\eta_0 \quad \text{as} \quad z\to+\infty, \tag{2.29}$$

where

$$\kappa_N = \int_{-\infty}^{\infty}((1+\mathrm{e}^{-2z'})^{-N/2}-H(z'))\,\mathrm{d}z',$$

$H(z')$ denoting the Heaviside step function. It therefore follows from (2.27) that z_0 is given by

$$z_0 = \frac{N}{2(N-2)}\ln \eta_0 + \frac{1}{(N-2)} + \kappa_N + x_0. \tag{2.30}$$

This result may also be obtained by using (2.15). We have thus established that

$$U(t) \sim a^{-N}(\tfrac{1}{2}M)^{N/(N-2)+N^2/(N-2)^2}t^{-N/(N-2)-N^3/2(N-2)^2}\exp(-N\eta_0 t^{N/(N-2)}) \quad \text{as} \quad t\to+\infty,$$

where the constant a is given by

$$a = (N-2)^{N/2(N-2)}\exp(1/(N-2)+\kappa_N+x_0), \tag{2.31}$$

so that $U(t)$ decays exponentially quickly for large t.

Returning to the original variables we therefore have at leading order as $t\to+\infty$

$$\left.\begin{aligned} u &\sim (M/2t)^{N/(N-2)}(a^2(\tfrac{1}{2}M)^{-2N/(N-2)^2}t^{N^2/(N-2)^2}\exp(2\eta_0 t^{N/(N-2)})+r^2)^{-N/2} \\ &\text{for} \quad r = O(t^{N^2/2(N-2)^2}\exp(\eta_0 t^{N/(N-2)})), \end{aligned}\right\} \tag{2.32}$$

with η_0 given by (2.22). Expression (2.32) may be rewritten in the unusual self-similar form

$$u \sim t^{-N/(N-2)-N^3/2(N-2)^2}\exp(-N\eta_0 t^{N/(N-2)})f(r/t^{N^2/2(N-2)^2}\exp(\eta_0 t^{N/(N-2)})). \tag{2.33}$$

In addition, (2.21) yields

$$u \sim \left(\frac{r^2\ln r}{(N-2)\,t}\right)^{-N/2} \quad \text{as} \quad t\to+\infty \quad \text{for} \quad \frac{\ln r}{t^{N/(N-2)}} > \eta_0. \tag{2.34}$$

A number of comments about this analysis are appropriate.

1. By making the rescaling

$$t\to M^{2/N}t, \quad r\to r, \quad u\to Mu,$$

we may without loss of generality set $M=1$. The manner in which M appears in (2.22) and (2.32) is consistent with this.

2. It follows from (2.12) and (2.31) that the constant a is proportional to r_0, the constant of proportionality depending only on N. We may without loss of generality also take $r_0 = 1$ by setting $\gamma = 1/r_0$ in the rescaling

$$t\to t, \quad r\to r/\gamma, \quad u\to\gamma^N u, \tag{2.35}$$

which leaves both (2.1) and the total mass (2.4) unchanged; in (2.32) $a\gamma$ then appears in place of a. When $n < 2/N$ only the total mass of the initial data, and not its spread, appears in the large-time behaviour (1.6) and (1.7); by contrast, in the case $n = 2/N$ the large-time behaviour (2.32) depends on r_0 as well as on M.

Under (2.32) the initial condition transforms to

$$\text{at} \quad t=0, \quad u = \gamma^{-N}I(r/\gamma); \tag{2.36}$$

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if the solution to (2.1) subject to (1.10) is

$$u = \bar{u}(r,t)$$

then the solution subject to (2.36) is therefore

$$u = \gamma^{-N}\,\bar{u}(r/\gamma, t). \tag{2.37}$$

In the limit in which $\gamma \to 0$ the initial condition (2.36) becomes $u = M\delta(r)/r^{N-1}$, and it is clear from (2.37) that no diffusion occurs in this limit. In terms of the formulation represented by (2.13), the change of variables (2.35) translates the initial conditions for $c(x,t)$ further and further to the left as γ decreases. For initial conditions for u in which only part of the total mass is contained in a delta function, that part remains immobile while the rest diffuses out.

We note that for general n the appropriate generalization of (2.35) which leaves (1.9) and (1.11) unchanged is

$$t \to \gamma^{nN-2}t, \quad r \to r/\gamma, \quad u \to \gamma^N u; \tag{2.38}$$

the existence of the instantaneous source similarity solution (1.6) for $n < 2/N$ corresponds to this invariance transformation.

3. While it does describe the appropriate large-time behaviour, the form (2.33) does not correspond to an exact similarity reduction of (2.1). It is, however, closely related to the group-invariant solution

$$u = \mathrm{e}^{-N\beta t} f(r/\mathrm{e}^{\beta t}),$$

where
$$f(\eta) = ((\beta/N)\,(a^2+\eta^2))^{-N/2},$$

and where β and a are arbitrary constants.

4. Assuming that the solution to (1.1) and (1.5) with $n = 2/N$ becomes radially symmetric as $t \to +\infty$, then the results of this section are also relevant to the case in which $I(\boldsymbol{x})$ is not radially symmetric.

5. As already noted, the form of the large-time behaviour depends on the nature of the initial conditions, which we assumed in the foregoing analysis to have compact support. To illustrate other possibilities we now consider the case in which

$$I(r) \sim Ar^{-b} \quad \text{as} \quad r \to \infty,$$

where A and b are positive constants with $b > N$ (so that (1.11) is finite). The behaviour as $t \to 0^+$ is now of the form

$$u \sim I(r) \qquad \text{for} \quad r = O(1),$$
$$u \sim t^{Nb/2(b-N)}\,\phi(\omega) \quad \text{for} \quad r = O(t^{-N/2(b-N)}),$$

where $\omega = rt^{N/2(b-N)}$ and $\phi(\omega)$ satisfies

$$\left.\begin{aligned} &\frac{N}{2(b-N)}\left(b\phi + \omega\frac{\mathrm{d}\phi}{\mathrm{d}\omega}\right) = \frac{1}{\omega^{N-1}}\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\omega^{N-1}\phi^{-2/N}\frac{\mathrm{d}\phi}{\mathrm{d}\omega}\right),\\ &\text{as} \quad \omega \to 0^+ \quad \phi \sim A\omega^{-b},\\ &\text{as} \quad \omega \to +\infty \quad \phi \to 0.\end{aligned}\right\} \tag{2.39}$$

It follows from (2.39) that

$$\phi^{-2/N} \sim \frac{\omega^2}{(N-2)}\left(\ln\omega - \frac{N(b-2)}{2(N-2)\,(b-N)}\ln\ln\omega - x_0 + o(1)\right) \quad \text{as} \quad \omega \to +\infty, \tag{2.40}$$

where x_0 is a constant which may be determined by solving (2.39); it follows by rescaling that

$$x_0 = (b-N)^{-1} \ln A + \mu_{N,b},$$

where the constant $\mu_{N,b}$ depends only on N and b.

The large-time expansion for $x = O(t^{N/(N-2)})$ again takes the form (2.19) and (2.21)–(2.24) remain valid. Using (2.40) it follows that in this case A_1 and A_2 are given by

$$A_1 = \frac{N^3}{4(N-2)^3} + \frac{N^2}{2(N-2)^2(b-N)}, \quad A_2 = \frac{Nx_0}{2(N-2)}.$$

The inner rescaling for $t \gg 1$ now takes the form

$$x = \eta_0 t^{2/(N-2)} + \left(\frac{N^2}{2(N-2)^2} + \frac{N}{(N-2)(b-N)}\right) \ln t + z. \tag{2.41}$$

The expansion (2.25) and the matching condition (2.26) and solution (2.28) remain valid, but the condition (2.27) is now replaced by

$$h_1 \sim -N(\eta_0/(N-2))^{-N/2} \left(z - \frac{N(b-2)}{2(N-2)(b-N)} \ln \eta_0 - x_0\right) \Big/ 2\eta_0 \quad \text{as} \quad z \to +\infty,$$

and it may then be shown that (2.30) becomes

$$z_0 = \frac{N(b-2)}{2(N-2)(b-N)} \ln \eta_0 + \frac{(b-2)}{(N-2)(b-N)} + \kappa_N + x_0.$$

The final result is that the large-time behaviour takes the self-similar form

$$u \sim t^{-N/(N-2)-N^3/2(N-2)^2-N^2/(N-2)(b-N)} \exp\left(-N\eta_0 t^{N/(N-2)}\right) f(\eta)$$

for
$$\eta = r/(t^{N^2/2(N-2)^2+N/(N-2)(b-N)} \exp(\eta_0 t^{N/(N-2)})) = O(1),$$

in place of (2.33). In the limit $b \to \infty$ we recover (2.33), as might be expected.

Hence, while the dominant (exponential) part of the time dependence is independent of b, the large-time behaviour does depend on b; this is again in contrast to the case $n < 2/N$ when the large-time behaviour (1.6) depends on the initial conditions only through M.

3. The limit $N \to \infty$

(a) *Introduction*

This section is concerned with the limit $N \to \infty$ which, while unphysical in this context, does provide valuable insight. We discuss the range $0 < n < 1$; the borderline case $n = 2/N$ has $n \ll 1$. For $n > 2/N$ there is a non-zero flux of material out to infinity, with

$$u \sim \left(\frac{1-n}{N-2} J(t)\right)^{1/(1-n)} r^{-(N-2)/(1-n)} \quad \text{as} \quad r \to \infty, \tag{3.1}$$

where $J(t)$, which must be determined as part of the solution, is the flux to infinity and is defined by

$$J(t) = \lim_{r \to \infty} \left(-r^{N-1} u^{-n} \frac{\partial u}{\partial r}\right).$$

For simplicity we shall in this section again discuss the case in which $I(r)$ has compact support, with

$$I(r) > 0 \quad \text{for} \quad r < r_0,$$
$$I(r) = 0 \quad \text{for} \quad r > r_0.$$

We shall need to consider the cases $n = O(1)$ and $n = O(1/N)$ separately. Equation (1.9) is more conveniently written in the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial r}\left(u^{-n}\frac{\partial u}{\partial r}\right) + \frac{(N-1)}{r}u^{-n}\frac{\partial u}{\partial r}, \tag{3.2}$$

and it is clear from (3.2) that we should introduce a rescaled time variable

$$\tau = Nt.$$

(*b*) $n = O(1)$, $0 < n < 1$

For $n = O(1)$ the asymptotic structure of the solution to (3.2) in the limit $N \to \infty$ comprises three regions. There are two outer regions, $r < s(\tau)$ (in which $u = O(1)$) and $r > s(\tau)$ (with u exponentially small), and an interior layer $r = s(\tau) + O(1/N)$; the location $s(\tau)$ of the interior layer is determined in the course of the analysis. The most significant feature of the solution is that it vanishes in finite time, and we define $\tau = \tau_c$ to be the extinction time, so that

$$u \equiv 0 \quad \text{for} \quad \tau \geqslant \tau_c.$$

We also introduce

$$\tau_{c0} = \lim_{N\to\infty} \tau_c.$$

Writing

$$u = u_0(r, \tau) + o(1) \quad \text{as} \quad N \to \infty \quad \text{for} \quad r < s(\tau)$$

we then obtain the first-order equation

$$\frac{\partial u_0}{\partial \tau} = \frac{1}{r}u_0^{-n}\frac{\partial u_0}{\partial r}, \tag{3.3}$$

which yields

$$u_0 = I((r^2 + 2u_0^{-n}\tau)^{\frac{1}{2}}). \tag{3.4}$$

The function $u_0(r, \tau)$ determined from (3.4) is in general multivalued and at any (r, τ) the larger value of u_0 is required. The required solution to (3.3) thus contains a shock, located at $r = s(\tau)$, say, with u_0 given by (3.4) for $r < s$ and with $u_0 = 0$ for $r > s$. The behaviour is illustrated schematically in figure 3. We assume that $I(r)$ is monotonically decreasing which ensures that only one shock occurs. The location $s(\tau)$ and structure of the shock may be determined by considering an interior layer with

$$r = s(\tau) + N^{-1}z,$$
$$u = \hat{u}_0(z, \tau) + o(1) \quad \text{as} \quad N \to \infty,$$

giving

$$-\dot{s}\frac{\partial \hat{u}_0}{\partial z} = \frac{\partial}{\partial z}\left(\hat{u}_0^{-n}\frac{\partial \hat{u}_0}{\partial z}\right) + \frac{1}{s}\hat{u}_0^{-n}\frac{\partial \hat{u}_0}{\partial z},$$

where

$$\dot{s} \equiv \mathrm{d}s/\mathrm{d}\tau.$$

Imposing $\hat{u}_0 \to 0$ as $z \to +\infty$, we obtain

$$\hat{u}_0 = \left(\exp\left(\frac{n}{(1-n)s}(z - z_0(\tau))\right) - (1-n)s\dot{s}\right)^{-1/n}, \tag{3.5}$$

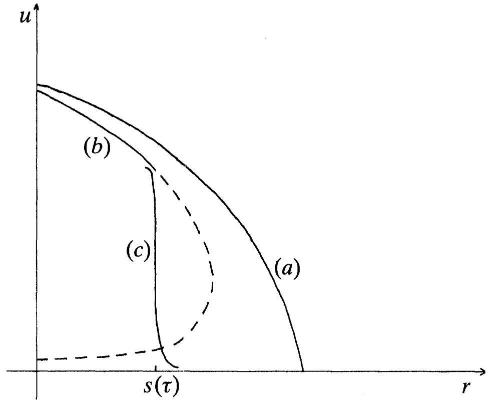

Figure 3. Asymptotic structure as $N \to \infty$ for $n = O(1)$. (a) $I(r)$. (b) u_0, see (3.4). (c) $\hat{u}_0$, see (3.5).

where $z_0(\tau)$ can be determined only by matching at higher orders. Matching (3.5) with (3.4) requires that

$$(1-n)\, s\dot{s} = -I^{-n}\,((s^2-2(1-n)\,\tau s\dot{s})^{\frac{1}{2}}), \tag{3.6}$$

which determines $s(\tau)$. The required initial condition on the ordinary differential equation (3.6) is that $s(0) = r_0$. We note that the expression (3.6) may alternatively be derived by writing (3.3) as the conservation law

$$\frac{\partial}{\partial \tau}(ru_0) - \frac{\partial}{\partial r}\left(\frac{1}{1-n} u_0^{1-n}\right) = 0,$$

which yields directly the shock condition

$$(1-n)\, s\dot{s} = -\lim_{r \to s^-} u_0^{-n}(r, \tau). \tag{3.7}$$

It is worth noting that this conservation law relates to the first moment, the result

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty ru\,\mathrm{d}r = -\frac{N-2}{1-n} u^{1-n}\,(0,t)$$

holding exactly (the integral is finite by (3.1)), and not to the mass.

For $r > s$ it follows from (3.5) that u is exponentially small in N. Writing

$$u = \psi^{1/(1-n)}$$

gives

$$\psi^{n/(1-n)} \frac{\partial \psi}{\partial \tau} = \frac{1}{N}\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\left(1-\frac{1}{N}\right)\frac{\partial \psi}{\partial r},$$

and applying the WKBJ method by writing

$$\ln \psi \sim -N\phi(r, \tau)$$

yields

$$\left(\frac{\partial \phi}{\partial r}\right)^2 = \frac{1}{r}\left(\frac{\partial \phi}{\partial r}\right). \tag{3.8}$$

Matching with (3.8) then gives

$$\phi = \ln\,(r/s).$$

This is consistent with the far-field expression (3.1).

The following points may be made about the preceding analysis.

1. It is valid only for $\tau < \tau_c$, the leading-order extinction time τ_{c0} being given by

$$s(\tau_{c0}) = 0.$$

2. The ordinary differential equation (3.6) can in principle be solved exactly for any given $I(r)$. Introducing $S = \frac{1}{2}s^2$ and $Q = -\dot{S}$, equation (3.6) may be rewritten as

$$S + (1-n)\,\tau Q = \Omega(Q), \tag{3.9}$$

where Ω is defined by

$$I^{-n}(((2\Omega(Q))^{\frac{1}{2}}) = (1-n)\,Q.$$

Differentiating (3.9) gives a linear equation for $\tau(Q)$:

$$nQ\,\mathrm{d}\tau/\mathrm{d}Q - (1-n)\,\tau = -\Omega'(Q).$$

Defining Q_0 by

$$\Omega(Q_0) = \tfrac{1}{2}r_0^2$$

we therefore have

$$\tau = \frac{Q^{(1-n)/n}}{n}\int_{Q_0}^{Q} \Omega'(\omega)\,\omega^{-1/n}\,\mathrm{d}\omega \tag{3.10}$$

and by (3.9)

$$S = \Omega(Q) - \frac{(1-n)\,Q^{1/n}}{n}\int_{Q_0}^{Q} \Omega'(\omega)\,\omega^{-1/n}\,\mathrm{d}\omega. \tag{3.11}$$

Equations (3.10) and (3.11) give the solution $S(\tau)$ in terms of the parameter Q. Defining $Q_c \equiv Q(\tau_{c0}), Q_c$ is given by the algebraic equation

$$n\Omega(Q_c)\,Q_c^{-1/n} = (1-n)\int_{Q_0}^{Q_c} \Omega'(\omega)\,\omega^{-1/n}\,\mathrm{d}\omega$$

and τ_{c0} may then be calculated from

$$\tau_{c0} = \Omega(Q_c)/(1-n)\,Q_c.$$

3. The behaviour close to $\tau = \tau_{c0}$ may be established as follows. Because

$$\dot{S} \sim -Q_c \quad \text{as} \quad \tau \to \tau_{c0}^-$$

we have

$$s(\tau) \sim (2Q_c(\tau_{c0}-\tau))^{\frac{1}{2}} \quad \text{as} \quad \tau \to \tau_{c0}^- \tag{3.12}$$

and, using (3.7), we find that

$$u_0 \sim ((1-n)\,Q_c)^{-1/n} \quad \text{as} \quad \tau \to \tau_{c0}^- \quad \text{for} \quad r < s(\tau). \tag{3.13}$$

(*c*) $n = O(1/N),\ n > 0$

On writing $u = v^{-1/n}$, it follows from (3.2) that

$$\frac{\partial v}{\partial t} = v\frac{\partial^2 v}{\partial r^2} - \frac{1}{n}\left(\frac{\partial v}{\partial r}\right)^2 + \frac{(N-1)}{r}\,v\frac{\partial v}{\partial r}. \tag{3.14}$$

In the limit $n \to 0$ with $N = O(1)$ a possible balance in (3.14) is given by

$$\frac{\partial v}{\partial t} \sim -\frac{1}{n}\left(\frac{\partial v}{\partial r}\right)^2,$$

which holds of $t = O(n)$. This corresponds to the formulation given by Kath & Cohen (1982) for $n \to 0$, $N = 1$. When N is large and n is small with $nN = O(1)$ a more complicated balance can occur in (3.14), namely

$$\frac{\partial v}{\partial \tau} \sim -\frac{1}{nN}\left(\frac{\partial v}{\partial r}\right)^2 + \frac{1}{r}v\frac{\partial v}{\partial r}, \tag{3.15}$$

and this corresponds to the case of interest here.

We consider the limit in which $I(r)$ is fixed as $n \to 0$, and we introduce

$$K = 1/nN.$$

The asymptotic structure in the limit $N \to \infty$ with $K = O(1)$ also contains three regions. There are again two outer regions, $r < s(\tau)$ (with $u = O(1)$) and $r > s(\tau)$ (where $v = O(1)$ with $v < 1$, so that u is exponentially small), and an interior layer $r = s(\tau) + O(1/N^{\frac{1}{2}})$.

For $r < s(\tau)$ it follows from (3.2) that at leading order we have

$$\frac{\partial u_0}{\partial \tau} = \frac{1}{r}\frac{\partial u_0}{\partial r}, \tag{3.16}$$

so that

$$u_0 = I((r^2 + 2\tau)^{\frac{1}{2}}). \tag{3.17}$$

No shock occurs in this case. The characteristics of (3.16) occupy the region

$$r^2 < r_0^2 - 2\tau,$$

and the location of the interior layer is given by

$$s(\tau) = (r_0^2 - 2\tau)^{\frac{1}{2}} \quad \text{for} \quad \tau < \tfrac{1}{2}r_0^2. \tag{3.18}$$

The behaviour in the interior layer depends on how $I(r)$ behaves close to $r = r_0$ and we again assume that

$$I(r) \sim A(r_0 - r)^b \quad \text{as} \quad r \to r_0^-.$$

The interior layer scalings are now

$$r = s(\tau) + N^{-\frac{1}{2}}z, \quad u = N^{-b/2}\hat{u};$$

we note that these inner scalings are different from those of §3*b*. At leading order for $\tau < \frac{1}{2}r_0^2$ we then have

$$\frac{\partial \hat{u}_0}{\partial \tau} = \frac{\partial^2 \hat{u}_0}{\partial z^2} - \frac{z}{s^2}\frac{\partial \hat{u}_0}{\partial z},$$

$$\text{as} \quad z \to -\infty, \quad \hat{u}_0 \sim A(-sz/r_0)^b,$$

$$\text{as} \quad z \to +\infty, \quad \hat{u}_0 \to 0,$$

$$\text{at} \quad \tau = 0, \quad \hat{u}_0 = A(-z)^b H(-z),$$

where we have matched with (3.17). In terms of the variables

$$Z = sz/r_0, \quad T = \tau - \tau^2/r_0^2$$

we have

$$\left.\begin{aligned} &\partial \hat{u}_0/\partial T = \partial^2 \hat{u}_0/\partial Z^2 \\ &\text{as} \quad Z \to -\infty, \quad \hat{u}_0 \sim A(-Z)^b, \\ &\text{as} \quad Z \to +\infty, \quad \hat{u}_0 \to 0, \\ &\text{at} \quad T = 0, \quad \hat{u}_0 = A(-Z)^b H(-Z). \end{aligned}\right\} \tag{3.19}$$

The solution to (3.19) takes the self-similar form

$$\hat{u}_0 = T^{b/2}\,\Psi(Z/T^{\frac{1}{2}});$$

more importantly, it follows from (3.19) that

$$\ln \hat{u}_0 \sim (r_0^2-2\tau)\,z^2/4(r_0^2-\tau)\,\tau \quad \text{as} \quad z\to+\infty, \tag{3.20}$$

so that $\hat{u}_0$ becomes exponentially small as $z\to+\infty$.

For $r > s(\tau)$ we have

$$\frac{\partial v_0}{\partial \tau} = \frac{1}{r}v_0\frac{\partial v_0}{\partial r} - K\left(\frac{\partial v_0}{\partial r}\right)^2. \tag{3.21}$$

To solve (3.21) we introduce $y = \frac{1}{2}r^2$ to give

$$\frac{\partial v_0}{\partial \tau} = v_0\frac{\partial v_0}{\partial y} - 2Ky\left(\frac{\partial v_0}{\partial y}\right)^2, \tag{3.22}$$

and Charpit's equations for the characteristics of this first-order partial differential equation can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = 4Kyp_0 - v_0, \quad \frac{\mathrm{d}v_0}{\mathrm{d}\tau} = 2Kyp_0^2, \quad \frac{\mathrm{d}p_0}{\mathrm{d}\tau} = -(2K-1)\,p_0^2, \tag{3.23}$$

where

$$p_0 = \partial v_0/\partial y.$$

The required initial conditions on (3.23) are

$$\text{at} \quad \tau = 0, \quad y = y_0, \quad v_0 = 1, \quad p_0 = P_0,$$

where $y_0 = \frac{1}{2}r_0^2$; the characteristics all emerge from the point $\tau = 0$, $y = y_0$ and are parametrized by P_0 with $0 \leqslant P_0 < \infty$. The resulting solution is easily derived in the form

$$p_0 = P_0/(1+(2K-1)\,P_0\,\tau),$$

with

$$K \neq \tfrac{1}{2}, \quad \left.\begin{aligned} y &= (y_0 + (2KP_0\,y_0 - 1)\,\tau))\,(1+(2K-1)\,P_0\,\tau)^{2K/(2K-1)}, \\ v_0 &= (1+(2KP_0\,y_0-1)\,P_0\,\tau)\,(1+(2K-1)\,P_0\,\tau)^{1/(2K-1)}; \end{aligned}\right\} \tag{3.24}$$

$$K = \tfrac{1}{2}, \quad \left.\begin{aligned} y &= (y_0 + (P_0\,y_0 - 1)\,\tau)\,\mathrm{e}^{P_0\tau}, \\ v_0 &= (1+(P_0\,y_0-1)\,P_0\,\tau)\,\mathrm{e}^{P_0\tau}. \end{aligned}\right\} \tag{3.25}$$

The solution $v_0(y,\tau)$ is given by eliminating P_0 from (3.24) or (3.25).

The characteristic projections of (3.22) take particularly simple forms for three particular values of P_0, as follows.

(i) $P_0 = 0$

It then follows from (3.24) that

$$y = y_0 - \tau, \tag{3.26}$$

with $v_0 = 1$ on this characteristic projection. The characteristic projection of (3.16) with $y = Y$ at $\tau = 0$ takes the form

$$y = Y - \tau, \tag{3.27}$$

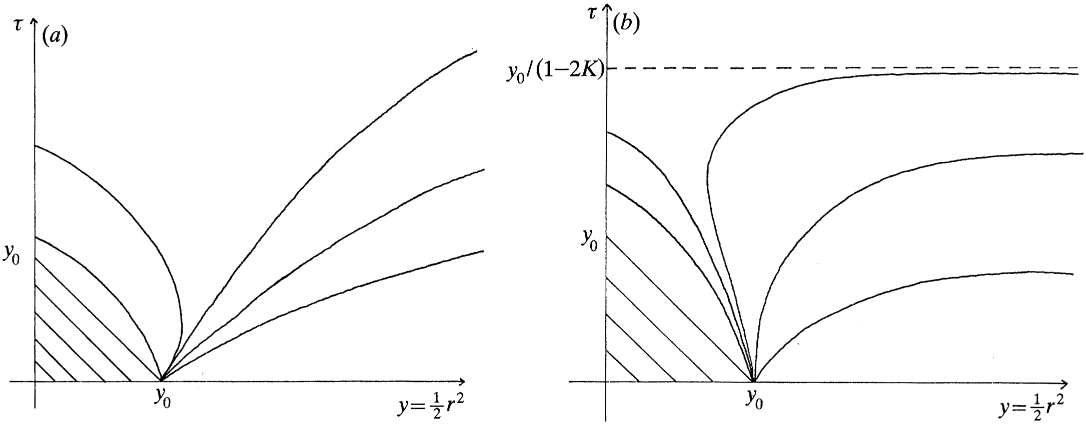

Figure 4. Characteristic projections of (3.16) and (3.21). (*a*) $n < 2/N$. (*b*) $n > 2/N$.

with $0 < Y \leqslant y_0$. For $Y = y_0$ this coincides with (3.26) and corresponds to $r = s(\tau)$; this is the location of the interior layer discussed above which describes the transition between (3.17) and (3.24). It is easily shown that (3.24) matches with (3.20).

(ii) $P_0 = 1/2Ky_0$.

In this case (3.24) gives

$$y = y_0(1+(2K-1)\,\tau/2Ky_0)^{2K/(2K-1)}. \tag{3.28}$$

(iii) $P_0 = 1/y_0$.

Then

$$y = y_0(1+(2K-1)\,\tau/y_0)^{(4K-1)/(2K-1)}. \tag{3.29}$$

At $\tau = y_0$ the region in which (3.17) holds disappears but the solutions (3.24) and (3.25) remain valid; it follows that to this order the details of the initial conditions are lost at $\tau = y_0$.

The solution (3.24) provides a simple means of exemplifying both the range $n < 2/N$ ($K > \frac{1}{2}$), in which mass is preserved for all time, and the range $n > 2/N$ ($K < \frac{1}{2}$) when solutions extinguish in finite time. The special case $K = \frac{1}{2}$ corresponds to the borderline case $n = 2/N$ considered in §2. We now discuss these three cases in turn.

(I) $K > \frac{1}{2}$. It follows from (3.24) that if $P_0 < 1/2Ky_0$ then the characteristic projection reaches $y = 0$ at $\tau = y_0/(1-2KP_0\,y_0)$. If $P_0 > 1/2Ky_0$ then we have

$$y \sim (2KP_0\,y_0-1)\,((2K-1)\,P_0)^{2K/(2K-1)}\,\tau^{(4K-1)/(2K-1)} \quad \text{as} \quad \tau \to +\infty.$$

The form of the characteristic projections is shown schematically in figure 4*a*. To determine the behaviour of v_0 as $\tau \to +\infty$ we must consider the characteristics with $P_0 - 1/2Ky_0 = O(1/\tau)$; (3.28) gives one of these. Using (3.24) we may then show that

$$v_0 \sim \tfrac{1}{2}\tau^{1/(2K-1)}\left(\left(\left(\frac{K}{2K-1}\right)^K r_0\right)^{-2/(2K-1)} + \frac{r^2}{(2K-1)\,\tau^{2K/(2K-1)}}\right)$$
$$\text{as} \quad \tau \to +\infty \quad \text{for} \quad r = O(\tau^{K/(2K-1)}). \tag{3.30}$$

This is equivalent to the instantaneous source solution (1.6) and (1.7). We note that the assumption that $I(r) = O(1)$ for $r < r_0$ implies that $M^{1/N} \sim r_0$; we can, however, set $M = O(1)$ by adopting suitable rescalings.

(II) $K < \frac{1}{2}$. It now follows from (3.24) that if $P_0 < 1/y_0$ then the characteristic reaches $y = 0$ at $\tau = y_0/(1-2KP_0 y_0)$; if $P_0 > 1/y_0$ we have $y \to \infty$ as $\tau \to 1/(1-2K)P_0$. We thus have leading-order extinction time $\tau_{c0} = y_0/(1-2K)$, the characteristics having all left the domain $0 \leqslant y < \infty$ by $\tau = \tau_{c0}$. A schematic of the characteristic projections for this case is given in figure 4*b*; this should be contrasted with figure 4*a*.

To determine the behaviour close to $\tau = \tau_{c0}$ we require characteristics on which $P_0 - 1/y_0 = O(\tau - \tau_{c0})$; the characteristic (3.29) is the one which provides the dividing line between those going to $y = 0$ and those on which $y \to \infty$. Defining ψ by

$$P_0 = (1 + (1-2K)(\tau - \tau_{c0})\psi/y_0)/y_0,$$

we find that as $\tau \to \tau_{c0}^-$

$$y \sim (1-2K-2K\psi)((1-2K)(1+\psi)/y_0)^{2K/(2K-1)}(\tau_{c0}-\tau)^{(4K-1)/(2K-1)},$$
$$v_0 \sim (1-2K+(1-4K)\psi)((1-2K)(1+\psi)/y_0)^{1/(2K-1)}(\tau_{c0}-\tau)^{2K/(2K-1)}/y_0,$$

which gives the solution parametrized by ψ; v_0 may therefore be expressed in the self-similar form

$$v_0 \sim (\tau_{c0}-\tau)^{-2K/(1-2K)} f(r/(\tau_{c0}-\tau)^{(1-4K)/(2(1-2K))}) \quad \text{as} \quad \tau \to \tau_{c0}^-. \tag{3.31}$$

In each of the cases (3.30) and (3.31) the asymptotic form is therefore self-similar, but the corresponding solutions are very different in nature.

(III) $K = \frac{1}{2}$. The solution is now given by (3.25) and the behaviour as $\tau \to +\infty$ is largely determined by the characteristics on which $P_0 - 1/y_0 = -1/\tau + O(1/\tau^2)$; we thus obtain

$$v_0 \sim \frac{r_0^2}{2\tau}\exp\left(\frac{2\tau}{r_0^2} - 1\right) + \frac{r^2}{r_0^2} \quad \text{as} \quad \tau \to +\infty \quad \text{for} \quad r = O(\tau^{-\frac{1}{2}}\exp(\tau/r_0^2)). \tag{3.32}$$

In addition, for $P_0 = O(1)$ with $P_0 > 1/y_0$ we have

$$v_0 \sim r^2 \ln r/\tau \quad \text{as} \quad \tau \to +\infty \quad \text{for} \quad \ln r/\tau > 1/r_0^2. \tag{3.33}$$

Expressions (3.32) and (3.33) should be compared with (2.32) and (2.34) respectively; because $M^{1/N} \sim r_0$ we have $\eta_0 \sim N/r_0^2$ as $N \to \infty$. The exponential part of the τ dependence in (3.32) is consistent with (2.32) but the power of τ appearing outside the exponential is different. This is because the limits $N \to \infty$ and $t \to \infty$ (just) fail to commute.

The following comments concern limiting cases of this analysis.

1. The limit $K \to \infty$ corresponds to the linear diffusion case $n = 0$. The exact solution to (1.9) and (1.10) for $n = 0$ may be written in the form

$$u = \frac{r^{-\nu}}{2t} e^{-r^2/4t} \int_0^\infty \rho^{\nu+1} e^{-\rho^2/4t} I_\nu\left(\frac{\rho r}{2t}\right) I(\rho)\, d\rho, \tag{3.34}$$

where $\nu = (N-2)/2$ and I_ν is the modified Bessel function. Applying standard asymptotic methods for integrals to (3.34) in the limit $N \to \infty$ produces results consistent with those given earlier.

2. The results of §3*b* in the limit $n \to 0^+$ are consistent with those of §3*c* in the limit $K \to 0^+$. For example, in the limit $n \to 0^+$ the expression (3.4) reproduces (3.17) and (3.6) yields (3.18). Writing $v_0 = \exp(\phi/K)$ in (3.21) and then taking the limit $K \to 0^+$ with $\phi > 0$ reproduces (3.8). The characteristic projections of (3.8) are lines on which

τ is constant; it follows from (3.24) that in the limit $K \to 0^+$ the characteristic projections of (3.21) are given for $P_0 > 1/y_0$ by

$$y \sim y_0 - \tau \quad \text{with} \quad 0 < \tau < 1/P_0,$$

and

$$\tau \sim 1/P_0 \quad \text{with} \quad y_0 - 1/P_0 < y < \infty.$$

3. We have shown that for $1 > n > 2/N$ the limiting behaviour close to the extinction time is self-similar for $N \to \infty$ (see (3.12), (3.13) and (3.31)). This form of behaviour will be the starting point of the analysis for $N = O(1)$ given in the next section.

4. $N > 2$ with $2/N < n < 1$

(a) *Introduction*

In this section we discuss equation (1.9) subject to (1.10) and (1.11) for $N > 2$ with $2/N < n < 1$. As already illustrated for the case $N \gg 1$, in this parameter range the solution extinguishes at some finite time, and we consider for the most part the behaviour close to the time of extinction. Since the solution to (1.1) and (1.5) is expected to become radially symmetric as the extinction time is approached, the analysis of this section is more generally applicable.

The behaviour as $r \to \infty$ takes the form (3.1), which in this parameter range is consistent with the requirement of finite total mass; $J(t)$ gives the flux of material to infinity, which must be determined as part of the solution. The total mass then satisfies

$$\int_0^\infty r^{N-1} u(r,t)\, \mathrm{d}r = M - \int_0^t J(t)\, \mathrm{d}t,$$

and extinction occurs at $t = t_c$, where

$$\int_0^{t_c} J(t)\, \mathrm{d}t = M.$$

The behaviour as $t \to t_c^-$ is self-similar with

$$u \sim (t_c - t)^\alpha f(r/(t_c-t)^\beta) \quad \text{as} \quad t \to t_c^-, \tag{4.1}$$

where α is given in terms of β by

$$\alpha = (1-2\beta)/n. \tag{4.2}$$

The value of β is determined from the problem

$$-\left(\alpha f - \beta\eta \frac{\mathrm{d}f}{\mathrm{d}\eta}\right) = \frac{1}{\eta^{N-1}} \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta^{N-1} f^{-n} \frac{\mathrm{d}f}{\mathrm{d}\eta}\right), \tag{4.3}$$

$$\left.\begin{aligned} &\text{at} \quad \eta = 0 \qquad \eta^{N-1} f^{-n} \frac{\mathrm{d}f}{\mathrm{d}\eta} = 0, \\ &\text{as} \quad \eta \to +\infty \quad f = O(\eta^{-(N-2)/(1-n)}), \end{aligned}\right\} \tag{4.4}$$

in which β plays the role of an eigenvalue. Because it is an eigenvalue problem, the solution to (4.3) and (4.4) contains an arbitrary constant; if $f = \bar{f}(\eta)$ is a solution then so is $f = \gamma^{-2/n} \bar{f}(\eta/\gamma)$ for any constant γ. We shall specify f uniquely by requiring that

$$\text{at} \quad \eta = 0, \quad f = 1, \tag{4.5}$$

in addition to (4.4). The problem may then be easily investigated numerically for particular parameter values since it may be solved as an initial-value problem from $\eta = 0$, and the value of β giving the required far-field behaviour can then be determined iteratively. Here we instead adopt approaches which provide analytical expressions for f and β in various special or limiting cases.

The equation (4.3) can be reformulated by writing

$$f = (\eta^2 h)^{-1/n}, \quad \xi = \ln \eta, \tag{4.6}$$

to give

$$h + \beta \frac{dh}{d\xi} = h\frac{d^2h}{d\xi^2} - \frac{1}{n}\left(\frac{dh}{d\xi}\right)^2 + \left(N + 2 - \frac{4}{n}\right) h \frac{dh}{d\xi} + 2\left(N - \frac{2}{n}\right) h^2, \tag{4.7}$$

which is amenable to phase plane methods. The formulation (4.7) will be useful in what follows.

We now consider a number of asymptotic limits together with a special case in which the solution to (4.3)–(4.5) may be determined exactly.

(*b*) $n \to (2/N)^+$

We write

$$n = 2/N + \epsilon;$$

it turns out that $\alpha, \beta = O(\epsilon^{-1})$ as $\epsilon \to 0$, and we anticipate this by writing

$$\hat{\alpha} = \epsilon\alpha, \quad \hat{\beta} = -\epsilon\beta$$

with

$$n\hat{\alpha} - 2\hat{\beta} = \epsilon.$$

In the limit $\epsilon \to 0$, (4.3)–(4.5) is a singular perturbation problem with inner region $\eta = O(\epsilon^{\frac{1}{2}})$ and outer region $\xi = O(\epsilon^{-1})$, ξ being given by (4.6).

Writing $\eta = \epsilon^{\frac{1}{2}}\hat{\eta}$, the leading-order inner problem is

$$-\hat{\beta}_0\left(Nf_0 + \hat{\eta}\frac{df_0}{d\hat{\eta}}\right) = \frac{1}{\hat{\eta}^{N-1}}\frac{d}{d\hat{\eta}}\left(\hat{\eta}^{N-1} f_0^{-2/N} \frac{df_0}{d\hat{\eta}}\right),$$

and imposing (4.5) gives

$$f_0 = (1 + (\hat{\beta}_0/N)\,\hat{\eta}^2)^{-N/2}. \tag{4.8}$$

An appropriate formulation for the outer problem is given by (4.7) with

$$\xi = \hat{\xi}/\epsilon, \quad h = \hat{h}/\epsilon$$

so that at leading order

$$\hat{h}_0 - \hat{\beta}_0\, d\hat{h}_0/d\hat{\xi} = -(N-2)\,\hat{h}_0\, d\hat{h}_0/d\hat{\xi} + N^2\hat{h}_0^2. \tag{4.9}$$

The matching condition on (4.9) which follows from (4.8) is

$$\text{at} \quad \hat{\xi} = 0, \quad \hat{h}_0 = \hat{\beta}_0/N. \tag{4.10}$$

Equation (4.9) can be rewritten as

$$\frac{d\hat{h}_0}{d\hat{\xi}} = \hat{h}_0\frac{(N^2\hat{h}_0 - 1)}{((N-2)\,\hat{h}_0 - \hat{\beta}_0)},$$

and, in order for the solution which satisfies (4.10) to have the required far-field behaviour, it follows that

$$\hat{\beta}_0 = (N-2)/N^2,$$

so that
$$\hat{h}_0 = \frac{(N-2)}{N^3} \exp\left(\frac{N^2}{N-2}\hat{\xi}\right).$$

We have therefore established that

$$\alpha \sim \frac{N-2}{nN-2}, \quad \beta \sim -\frac{(N-2)}{N(nN-2)} \quad \text{as} \quad nN-2 \to 0^+,$$

so that the decay of u is very slow close to the extinction time.

(*c*) $n \to 1^-$

We note that in terms of the model (1.2)–(1.4) this limit has a specific physical interpretation, namely that the clusters are very large. A value $m = 12$ has been proposed for the diffusion of boron in silicon by Ryssel *et al*. (1980).

In this limit we have $\alpha \ll 1$, and we write

$$n = 1-\epsilon,$$
$$\alpha = \delta, \quad \beta = \tfrac{1}{2}(1-(1-\epsilon)\,\delta),$$

with $\epsilon \ll 1$ and where $\delta(\epsilon) \ll 1$ remains to be determined.

In (4.3) we write $f = 1+\delta F$ for $\eta = O(1)$, so that at leading order

$$-1+\tfrac{1}{2}\eta\frac{dF_0}{d\eta} = \frac{1}{\eta^{N-1}}\frac{d}{d\eta}\left(\eta^{N-1}\frac{dF_0}{d\eta}\right),$$

giving

$$F_0 = -\int_0^{\eta} \eta'^{1-N}\, e^{\frac{1}{4}\eta'^2} \int_0^{\eta'} \eta''^{N-1}\, e^{-\frac{1}{4}\eta''^2}\, d\eta''\, d\eta';$$

it follows that

$$F_0 \sim -2k_N\, \eta^{-N}\, e^{\frac{1}{4}\eta^2} \quad \text{as} \quad \eta \to +\infty, \tag{4.11}$$

where
$$k_N = 2^{N-1}\,\Gamma(\tfrac{1}{2}N).$$

It is evident from (4.11) that F_0 is of $O(1/\delta)$ in an asymptotically narrow region

$$\eta = \eta_0(\delta)\, \ln^{\frac{1}{2}}(1/\delta) + z/\ln^{\frac{1}{2}}(1/\delta), \tag{4.12}$$

where $\eta_0(\delta)$ is defined by

$$\delta^{\frac{1}{4}\eta_0^2} - 1 = 2k_N\, \eta_0^{-N}\, \ln^{-N/2}(1/\delta), \tag{4.13}$$

so that
$$\eta_0 \sim 2 \quad \text{as} \quad \delta \to 0. \tag{4.14}$$

We thus have a transition layer with $z = O(1)$ in which we write

$$f \sim f_0(z) \quad \text{as} \quad \delta \to 0,$$

and the leading-order matching condition

$$f_0 \sim 1-e^z \quad \text{as} \quad z \to -\infty \tag{4.15}$$

then follows from (4.11) and (4.13). Using (4.12) and (4.14) we obtain

$$\frac{df_0}{dz} = \frac{d}{dz}\left(f_0^{-1}\frac{df_0}{dz}\right),$$

so imposing (4.15) gives

$$f_0 = 1/(1+e^z). \tag{4.16}$$

The final region we have to discuss is $\eta = O(\ln^{\frac{1}{2}}(1/\delta))$ with $\eta/\ln^{\frac{1}{2}}(1/\delta) > 2$. We write

$$\eta = \ln^{\frac{1}{2}}(1/\delta)\,\eta^{\dagger}, \quad g = -\ln f/\ln(1/\delta),$$

to give at leading order

$$\frac{\mathrm{d}}{\mathrm{d}\eta^{\dagger}}\left(\eta^{\dagger N-1}\exp\left(-\epsilon\ln(1/\delta)\,g_0\right)\frac{\mathrm{d}g_0}{\mathrm{d}\eta^{\dagger}}\right) = 0, \tag{4.17}$$

with matching condition

$$g_0 \sim \eta^{\dagger} - 2 \quad \text{as} \quad \eta^{\dagger} \to 2^{+}, \tag{4.18}$$

which follows from (4.16). Equation (4.17) implies that $\epsilon = O(1/\ln(1/\delta))$. The far-field condition requires that $g_0 \to +\infty$ as $\eta^{\dagger} \to +\infty$, so that

$$g_0 = \frac{N-2}{\epsilon\ln(1/\delta)}\ln(\eta^{\dagger}/A),$$

where A is a constant. Matching with (4.18) then yields

$$A = 2, \quad \epsilon\ln(1/\delta) = \tfrac{1}{2}(N-2),$$

so that

$$g_0 = 2\ln(\tfrac{1}{2}\eta^{\dagger}).$$

Hence as $n \to 1^{-}$

$$\ln(1/\alpha) \sim (N-2)/2(1-n)$$

so that α is exponentially small, and the approach to extinction is very rapid.

(d) $n = 4/(N+2)$

This is a case in which exact results can be derived. In King (1992*d*) exact solutions to

$$\frac{\partial u}{\partial t} = \frac{1}{r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}u^{-4/(N+2)}\frac{\partial u}{\partial r}\right)$$

of the form

$$u = (a_0(t) + a_1(t)\,r^2 + a_2(t)\,r^4)^{-(N+2)/4} \tag{4.19}$$

were constructed. These solutions have finite mass and extinguish in finite time with

$$u \sim (t_c - t)^{(N+2)/4} f(r) \quad \text{as} \quad t \to t_c^{-}. \tag{4.20}$$

We infer that (4.20) describes the asymptotic form of solutions for arbitrary initial conditions. Taking $\alpha = \frac{1}{4}(N+2)$, $\beta = 0$, the required solution to (4.3)–(4.5) then takes the form

$$f(\eta) = \left(1 + \frac{1}{2N}\eta^2 + \frac{1}{16N^2}\eta^4\right)^{-(N+2)/4}.$$

This special case is of particular interest because it provides the dividing line between the case

$$\beta < 0 \quad n < 4/(N+2),$$

in which the solution (4.1) spreads out to infinity as $t \to t_c^{-}$ and the case

$$\beta > 0 \quad n > 4/(N+2),$$

in which the range of (4.1) contracts to zero as $t \to t_c^{-}$.

(e) *Other results*

We believe on the basis of a phase plane analysis of (4.7) that, for each n and N, (4.3) and (4.4) admits a non-trivial solution for exactly one value of β, which we denote here as $\beta^*(n,N)$. In this section we note some results which are relevant to such an analysis, without going into all of the details.

If we define

$$p = \mathrm{d}h/\mathrm{d}\xi,$$

then the required solution to (4.7) must satisfy

$$p \sim \frac{nN-2}{1-n}h \quad \text{with} \quad h \to +\infty \quad \text{as} \quad \xi \to +\infty, \tag{4.21}$$

$$p \sim -2h \quad \text{with} \quad h \to +\infty \quad \text{as} \quad \xi \to -\infty. \tag{4.22}$$

For each value of β, there is only one trajectory with the behaviour (4.21) and one with the behaviour (4.22). Only for $\beta = \beta^*$ does the same trajectory satisfy both (4.21) and (4.22).

We now note four exact solutions of (4.3) for particular values of β which correspond to straight line trajectories in the phase plane; in each case A is an arbitrary constant.

$$\text{(i)}\quad \beta = -\frac{1}{nN-2}, \quad f = \left(A + \frac{n\eta^2}{2(nN-2)}\right)^{-1/n}; \qquad p = -2h + \frac{n}{nN-2}. \tag{4.23}$$

$$\text{(ii)}\quad \beta = -\frac{1-n}{nN-2}, \quad f = A\eta^{-(N-2)/(1-n)}; \qquad p = \frac{nN-2}{1-n}h. \tag{4.24}$$

$$\text{(iii)}\quad \beta = \tfrac{1}{2}, \quad f = A; \qquad p = -2h. \tag{4.25}$$

$$\text{(iv)}\quad \beta = \frac{1}{2(1-n)}, \quad f = \left(A\eta^{n(N-2)/(1-n)} + \frac{n\eta^2}{2(nN-2)}\right)^{-1/n}; \quad p = \frac{nN-2}{1-n}h - \frac{n}{2(1-n)}. \tag{4.26}$$

Because these trajectories each satisfy either (4.21) or (4.22), they can be used in conjunction with the phase plane analysis to obtain bounds on β^*; in particular we have

$$-(1-n)/(nN-2) < \beta^* < \tfrac{1}{2}.$$

The formulation (4.7) also leads to the following observation, which corresponds to a special case of the transformations discussed in King (1992*a*). If we introduce the change of variables

$$h = \mu^2 h', \quad \xi = \mu\xi', \tag{4.27}$$

with $\mu = -2(1-n)/(nN-2)$, then we obtain

$$h' + \beta'\frac{\mathrm{d}h'}{\mathrm{d}\xi'} = h'\frac{\mathrm{d}^2h'}{\mathrm{d}\xi'^2} - \frac{1}{n}\frac{\mathrm{d}h'}{\mathrm{d}\xi'} + \left(N' + 2 - \frac{4}{n}\right)h'\frac{\mathrm{d}h'}{\mathrm{d}\xi'} + 2\left(N' - \frac{2}{n}\right)h'^2, \tag{4.28}$$

where

$$N' = 2(N-4+2n)/(nN-2), \quad \beta' = -(nN-2)\beta/2(1-n). \tag{4.29}$$

The equation (4.28) is of the same form as (4.7), and it is then readily seen that β^* satisfies

$$\beta^*(n, 2(N-4+2n)/(nN-2)) = -(nN-2)\,\beta^*(n,N)/2(1-n). \tag{4.30}$$

The transformation (4.27) maps the cases $4/(N+2) < n < 1$ and $2/N < n < 4/(N+2)$ into one another. For $n = 4/(N+2)$ we have $N = N'$ and the result $\beta^* = 0$ for $n = 4/(N+2)$ then follows immediately from (4.30).

(*f*) *Discussion*

A number of remarks may be made about the results of this section.

1. The exponent β in (4.1) is determined from the nonlinear eigenvalue problem (4.3). The similarity solution (4.1) is thus one of the second kind in the sense of Barenblatt (1979). Similarity solutions of the second kind have also recently occurred in certain other nonlinear diffusion problems (see, for example, Aronson *et al.* 1992; Bernis *et al.* 1992).

2. The limit $N \to \infty$ in (4.3) can also be considered and the results obtained are consistent with those of §3. Specifically we may show that, for $2/N < n < 1$, we have

$$\alpha \sim \frac{2}{n(nN-2)}, \quad \beta \sim \frac{nN-4}{2(nN-2)} \quad \text{as} \quad N \to \infty \quad \text{for} \quad nN = O(1)$$

and

$$\ln(1/\alpha) \sim \tfrac{1}{2}N(n/(1-n) + \ln(1-n)) \quad \text{as} \quad N \to \infty \quad \text{for} \quad n = O(1).$$

In the latter case α is again exponentially small; this result is a refinement of those of §3*b* close to the extinction time.

3. If $u = \bar{u}(r,t)$ is the solution for the initial condition

$$\text{at} \quad t = 0, \quad u = I(r),$$

and the corresponding extinction time is $t = t_c$, then using the rescaling (2.38) the solution for the initial condition

$$\text{at} \quad t = 0, \quad u = \gamma^{-N} I(r/\gamma), \tag{4.31}$$

is given by

$$u = \gamma^{-N}\,\bar{u}(r/\gamma, \gamma^{nN-2}\,t)$$

and the extinction time becomes

$$t = \gamma^{-(nN-2)}\,t_c.$$

In the limit in which $\gamma \to 0$ in (4.31) we obtain delta function initial conditions; in this limit the extinction time becomes infinite and no diffusion occurs. The issue of delta function initial conditions is addressed rigorously in Brezis & Friedman (1983).

4. Our results indicate that

$$u \sim \gamma^{-2/n}(t_c - t)^{(1-2\beta)/n} f(r/\gamma(t_c-t)^{\beta}) \quad \text{as} \quad t \to t_c^-, \tag{4.32}$$

where f and β are determined uniquely by (4.3)–(4.5). The asymptotic behaviour (4.32) depends on the initial conditions only through the constants γ and t_c, which cannot be determined from an analysis of the limit $t \to t_c^-$. The dependence of t_c and γ on the initial conditions is an interesting open question. In general these quantities can presumably only be determined numerically, but the following asymptotic results can be obtained by determining the evolution over earlier times:

$$n - 2/N = \epsilon, \quad t_c \sim \epsilon^{-(N-2)/N}\,T_c, \quad \ln\gamma \sim (N-2)\ln((N-2)\,T_c/N)/N^2\epsilon \quad \text{as} \quad \epsilon \to 0^+,$$

where $T_c = N^{-2(N-2)/N}(\frac{1}{2}M)^{2/N}$ (the analysis of this case relies on that of §2);

$$n = 1-\epsilon, \quad t_c \sim \epsilon \int_0^\infty rI(r)\,dr/(N-2), \quad \gamma \sim (I(0))^{-\frac{1}{2}} \quad \text{as} \quad \epsilon \to 0^+.$$

5. Self-similar behaviour for (1.1) close to the extinction time has also been discussed by Galaktionov & Posashkov (1986), but there are a number of important differences from the results given here. Galaktionov & Posashkov (1986) consider only similarity solutions of infinite mass, mainly those which behave as

$$u \sim (nr^2/2(nN-2)\,(t^*-t))^{-1/n} \quad \text{as} \quad r \to \infty, \tag{4.33}$$

where t^* is a constant. They show that similarity solutions of the form

$$u = (t^*-t)^{(1-2\beta)/n} f(r/(t^*-t)^\beta) \tag{4.34}$$

satisfying (4.33) exist for $\beta \leqslant -1/(nN-2)$ (in fact it follows from the phase plane analysis that such solutions exist for

$$\beta < \min(\beta_c, \beta^*),$$

where $\beta_c = (nN+2n-4)/2(nN-2)$ is the value of β at which the relevant critical point changes stability).

Such solutions require that the initial conditions behave as

$$I \sim Ar^{-2/n} \quad \text{as} \quad r \to \infty \tag{4.35}$$

and it follows from (4.33) that

$$t^* = nA^n/(nN-2). \tag{4.36}$$

For initial conditions of the form (4.35), the extinction time t_c may be given by $t_c = t^*$ (as illustrated by solutions of the form (4.34) which vanish at $t = t^*$) or may satisfy $t_c > t^*$. In the latter case, at $t = t^*$ the far-field behaviour changes from (4.33) to the finite mass form (3.1) with $J(t) \to +\infty$ as $t \to t^{*+}$. This behaviour may be illustrated by similarity solutions of the form

$$\left.\begin{aligned} u &= (t^*-t)^{\gamma/(n\gamma-2)} f(r(t^*-t)^{1/(n\gamma-2)}), & t &< t^*, \\ u &= Cr^{-\gamma}, & t &= t^*, \\ u &= (t-t^*)^{\gamma/(n\gamma-2)}\, g(r(t-t^*)^{1/(n\gamma-2)}) & t &> t^*, \end{aligned}\right\} \tag{4.37}$$

which can describe the far-field behaviour of more general solutions for t close to t^*. In (4.37), C and γ are positive constants with

$$(N-2)/(1-n) > \gamma > 2/n$$

and f and g satisfy

$$\begin{aligned} f(\eta), g(\eta) &\sim C\eta^{-\gamma} & &\text{as} \quad \eta \to 0^+, \\ f(\eta) &\sim (n\eta^2/2(nN-2))^{-1/n} & &\text{as} \quad \eta \to +\infty, \\ g(\eta) &\sim D\eta^{-(N-2)/(1-n)} & &\text{as} \quad \eta \to +\infty, \end{aligned}$$

for some constant D. The existence of such similarity solutions may again be established via the phase plane.

For initial conditions which satisfy (4.35), various types of behaviour close to $t = t^*$ are therefore possible. Which actually occurs is expected to be determined by the precise behaviour of the initial conditions for large r. When $t_c = t^*$ and (4.34) describes the behaviour close to extinction, the similarity exponent depends on the initial conditions and the extinction time can be directly determined from the initial conditions by (4.36). By contrast, in the finite mass case discussed earlier (and for many infinite mass initial conditions decaying more rapidly than (4.35) as $r \to \infty$) the similarity exponent is independent of the initial conditions, being determined from (4.3) and (4.4), but the extinction time cannot in general be calculated exactly.

In summary, the similarity solutions of the form (4.34) discussed by Galaktionov & Posashkov (1986) are applicable only to certain infinite mass initial conditions. The important role played by similarity solutions of the second kind was not noted by these authors.

5. $n = 1$, $N = 2$

The two-dimensional case in which $n = 1$ is remarkable in a number of respects. The equation we are considering is

$$\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r u^{-1}\frac{\partial u}{\partial r}\right), \tag{5.1}$$

for which several applications are known. We note in particular the work of de Gennes (1984) on the spreading of microscopic droplets. It is suggested in de Gennes (1984) that the similarity solution

$$u = \mathrm{e}^{-2\alpha t} f(r/\mathrm{e}^{\alpha t}), \tag{5.2}$$

with
$$f(\eta) = 2/\alpha(a^2+\eta^2)$$

and where α and a are constants, may be of physical relevance even though it corresponds to infinite total mass. We shall indicate that it is not in fact relevant to the solution to (5.1) subject to (1.10) with finite mass M.

We start by noting possible far-field balances for (5.1) since, as in §2, such behaviour is crucial in determining asymptotic forms of solution. There are two possibilities.

(a) *Quasi-steady balance*

$$u \sim A(t)\, r^{-J(t)} \quad \text{as} \quad r \to \infty \quad \text{with} \quad J > 2. \tag{5.3}$$

The functions A and J are arbitrary, subject to $J > 2$; higher-order terms can easily be constructed, giving

$$u \sim A r^{-J}\left(1 - \frac{A\dot{J}}{(J-2)^2} r^{-(J-2)} \ln r + \frac{1}{(J-2)^2}\left(\dot{A} - \frac{2A\dot{J}}{(J-2)}\right) r^{-(J-2)}\right) \quad \text{as} \quad r \to \infty,$$

where $\dot{}$ denotes $\mathrm{d}/\mathrm{d}t$. For this form of solution the flux satisfies

$$-r u^{-1} \partial u/\partial r \to J(t) \quad \text{as} \quad r \to \infty$$

so that
$$\int_0^\infty r u \,\mathrm{d}r = M - \int_0^t J(t)\,\mathrm{d}t. \tag{5.4}$$

(b) *Separable balance*

$$u \sim 2t/r^2 \ln^2 r \quad \text{as} \quad r \to \infty. \tag{5.5}$$

As in §2, we may seek an asymptotic form

$$u \sim tF(r) \quad \text{as} \quad r \to \infty. \tag{5.6}$$

The general solution to (5.1) of the form (5.6) may be written

$$F(r) = 2\gamma^2 r_0^\gamma / r^{2+\gamma} (1-(r/r_0)^{-\gamma})^2,$$

where r_0 and γ are arbitrary constants; the behaviour of this solution in the limit $r \to \infty$ is a special case of (5.3). However, corresponding to the limit $\gamma \to 0$ we also have a singular solution

$$F(r) = 2/r^2 \ln^2 (r/r_0), \tag{5.7}$$

which produces the required asymptotic behaviour (5.5).

It follows from (5.5) that

$$-ru^{-1}\,\partial u/\partial r \to 2 \quad \text{as} \quad r \to \infty, \tag{5.8}$$

so that

$$\int_0^\infty ru\,\mathrm{d}r = M - 2t. \tag{5.9}$$

By comparison with (5.4) we note that this represents the minimum rate of loss of material to infinity (because $J > 2$). It is evident that expressions (5.4) and (5.9) both imply finite-time extinction with $u \equiv 0$ for $t \geqslant t_c$. In the case of (5.9) we have

$$t_c = \tfrac{1}{2}M.$$

The situation may be further clarified by introducing the change of variables

$$u = r^{-2} c, \quad x = \ln r,$$

to give, as in King (1992*a*), the one-dimensional equation

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(c^{-1}\frac{\partial c}{\partial x}\right). \tag{5.10}$$

Conditions on (5.10) are that

$$c \sim U(t)\,\mathrm{e}^{2x} \quad \text{as} \quad x \to -\infty, \tag{5.11}$$

where $U(t) = u(0,t)$ must be determined as part of the solution, and

$$c \sim A(t)\,\mathrm{e}^{-(J(t)-2)x} \quad \text{as} \quad x \to +\infty, \tag{5.12}$$

when (5.2) holds, while

$$c \sim 2t/x^2 \quad \text{as} \quad x \to +\infty \tag{5.13}$$

corresponds to (5.5). The one-dimensional problems (5.10) and (5.11) with (5.12) or (5.13) are of the type discussed in Rodriguez & Vazquez (1990) and the required solution to (5.10) is non-maximal in the sense discussed there. It follows from the results of Rodriguez & Vazquez (1990) that, in order for the solution to (5.10)–(5.12) with suitable initial conditions to be specified uniquely, $J(t)$ must be prescribed; $A(t)$ is then determined as part of the solution. Because $J(t)$ must be specified beforehand, it is possible to explicitly calculate the extinction time from (5.4); we have

$$\int_0^{t_c} J(t)\,\mathrm{d}r = M.$$

The case $n = 1$, $N = 2$ therefore shares the property of the cases $1 \leqslant n < 2$, $N = 1$ that the solution to (1.9) and (1.10) is not uniquely specified unless the flux to infinity is also prescribed. An important difference, however, is that for $1 \leqslant n < 2, N = 1$ the maximal solution preserves mass whereas for $n = 1, N = 2$ it does not, the minimum flux to infinity being given by (5.8).

As we shall indicate later, the most important solution physically is likely to be that in which (5.5) holds. However, before discussing this case we make some brief comments concerning solutions for which (5.3) holds.

1. Equation (5.10) admits similarity solution of the form

$$c = (t_c - t)\, g(x + \mu \ln (t_c - t)), \tag{5.14}$$

where μ is an arbitrary constant and $g(\zeta)$ satisfies

$$-g - \mu \frac{dg}{d\zeta} = \frac{d}{d\zeta}\left(g^{-1}\frac{dg}{d\zeta}\right).$$

Writing $g = e^h$ and integrating gives

$$\left.\begin{aligned} \mu \neq 0 \quad & 2\mu - \ln(1+2\mu) - \mu^2 e^h = \mu\, dh/d\zeta - \ln(1+\mu\, dh/d\zeta), \\ \mu = 0 \quad & 2 - e^h = \tfrac{1}{2}(dh/d\zeta)^2, \end{aligned}\right\} \tag{5.15}$$

where we have imposed the condition (5.11). It follows from (5.15) that as $\zeta \to +\infty$ we have

$$h \sim \nu\zeta,$$

where ν is the positive root of

$$2\mu - \ln(1+2\mu) = -\mu\nu - \ln(1-\mu\nu). \tag{5.16}$$

The solution (5.14) describes the behaviour at $t \to t_c^-$ with

$$\nu = J(t_c) - 2, \tag{5.17}$$

(see (5.12)) and μ can then be determined from (5.16). In particular we have

$$\left.\begin{aligned} &\nu \to +\infty, && \mu \sim -\tfrac{1}{2} + \tfrac{1}{4}(\nu+2)\, e^{-\frac{1}{2}(\nu+2)}, \\ &\nu = 2, && \mu = 0, \\ &\nu \to 0^+, && \mu \sim 4/\nu^2. \end{aligned}\right\} \tag{5.18}$$

It is evident that the behaviour is singular in the limit $\nu \to 0$ which corresponds to (5.13); the case (5.13) must therefore be discussed separately.

In the original variables the solution (5.14) takes the self-similar form

$$u = (t_c - t)^{1+2\mu} f(r(t_c - t)^{\mu}).$$

We note that the behaviour close to extinction is therefore again given by a similarity solution of the second kind; in this case the eigenvalue μ can be obtained exactly from (5.16) with ν given by (5.17).

2. Solutions of the form (4.18) exist whenever $n = 4/(N+2)$, so (5.1) admits solutions with

$$u = 1/(a_0(t) + a_1(t)\, r^2 + a_2(t)\, r^4).$$

Such solutions have

$$-ru^{-1}\,\partial u/\partial r \to 4 \quad \text{as} \quad r \to \infty$$

and their behaviour close to extinction takes the form

$$u = (t_c - t) f(r),$$

in agreement with (5.18). The results of §4 indicate that the values $n = 1$, $n = 2/N$ and $n = 4/(N+2)$ are all special cases and it is worth noting that these three conditions are simultaneously satisfied when $n = 1$, $N = 2$.

We now turn to the case of the 'maximal' solution in which (5.10) is to be solved subject to (5.11) and (5.13) with

$$\int_{-\infty}^{\infty} c\,\mathrm{d}x = M - 2t. \tag{5.19}$$

We again require a more detailed description of the behaviour as $x \to +\infty$ and, once more treating the case of compact support with

$$I(r) \sim A(r_0 - r)^b \quad \text{as} \quad r \to r_0^-,$$

we have for $t \ll 1$

$$u \sim I(r), \qquad r < r_0,$$
$$u \sim t^{b/(b+2)}\,\phi(\omega), \qquad r = r_0 + O(t^{1/(b+2)}),$$
$$u \sim tF(r), \qquad r > r_0,$$

where $\omega = (r - r_0)\,t^{-1/(b+2)}$ and $\phi(\omega)$ is given by (2.7) with $N = 2$ and where $F(r)$ is given by (5.7).

It can be shown that the condition

$$u = \frac{2t}{r^2 \ln^2 r}\left(1 + \frac{2 \ln r_0}{\ln r} + o\left(\frac{1}{\ln r}\right)\right) \quad \text{as} \quad r \to \infty$$

then holds for all t so, defining x_0 by

$$x_0 = \ln r_0,$$

we have

$$c \sim (2t/x^2)(1 + 2x_0/x) \quad \text{as} \quad x \to +\infty. \tag{5.20}$$

To determine the behaviour close to the extinction time $t_c = \tfrac{1}{2}M$ we write

$$t = t_c + T$$

with $T < 0$. Expressions (5.13) and (5.19) then give

$$c \sim M/x^2 \quad \text{as} \quad x \to +\infty, \quad T \to 0^-, \tag{5.21}$$

with
$$\int_{-\infty}^{\infty} c\,\mathrm{d}x = -2T. \tag{5.22}$$

If we seek a self-similar form consistent with these two conditions then we need

$$c \sim (-T)^2 g_0(\eta) \quad \text{as} \quad T \to 0^-, \tag{5.23}$$

where $\eta = x(-T)$, and the leading-order balance in (5.10) is simply

$$-2g_0 - \eta\,\mathrm{d}g_0/\mathrm{d}\eta = 0,$$

so it follows from (5.21) that

$$g_0 = M/\eta^2. \tag{5.24}$$

This expression is evidently not valid for all η and an inner region is again needed; the result (5.23) is valid for

$$x = O(1/(-T)) \quad \text{as} \quad T \to 0^- \quad \text{with} \quad x(-T) > \eta_0,$$

where η_0 is given from (5.22) by

$$\int_{\eta_0}^{\infty} g_0(\eta)\,\mathrm{d}\eta = 2$$

so that
$$\eta_0 = \tfrac{1}{2}M. \tag{5.25}$$

We again need a correction term; writing

$$c \sim (-T)^2 g_0(\eta) + (-T)^3 g_1(\eta) \quad \text{as} \quad T \to 0^- \tag{5.26}$$

yields
$$-3g_1 - \eta \frac{\mathrm{d}g_1}{\mathrm{d}\eta} = \frac{\mathrm{d}}{\mathrm{d}\eta}\left(g_0^{-1}\frac{\mathrm{d}g_0}{\mathrm{d}\eta}\right)$$

so that
$$g_1 = (-2 + A_1/\eta)/\eta^2, \tag{5.27}$$

where it follows from (5.20) that

$$A_1 = 2Mx_0.$$

Using (5.25), the inner region has scaling

$$x = M/2(-T) + z$$

with
$$c \sim (-T)^2 h_0(z) + (-T)^3 h_1(z) \quad \text{as} \quad T \to 0^-$$

so that
$$-\tfrac{1}{2}M\frac{\mathrm{d}h_0}{\mathrm{d}z} = \frac{\mathrm{d}}{\mathrm{d}z}\left(h_0^{-1}\frac{\mathrm{d}h_0}{\mathrm{d}z}\right), \tag{5.28}$$

and
$$-2h_0 - \tfrac{1}{2}M\frac{\mathrm{d}h_1}{\mathrm{d}z} = \frac{\mathrm{d}^2}{\mathrm{d}z^2}(h_0^{-1}h_1). \tag{5.29}$$

Matching with (5.26) implies that

$$\left.\begin{aligned} h_0 &\sim 4/M && \text{as} \quad z \to +\infty, \\ h_1 &\sim -8(2z + 1 - 2x_0)/M^2 && \text{as} \quad z \to +\infty. \end{aligned}\right\} \tag{5.30}$$

In $x(-T) < \tfrac{1}{2}M$ we find using (5.11) that

$$\ln c - 2x + U(t_c + T)$$

is exponentially small in $-T$ as $T \to 0^-$ and we therefore also require that

$$\ln h_0,\ \ln h_1 \sim 2z \quad \text{as} \quad z \to -\infty.$$

Equation (5.28) thus gives
$$h_0 = 4/M(1 + \mathrm{e}^{-2(z - z_0)}) \tag{5.31}$$
where the constant z_0 remains to be determined, and from (5.29) we then obtain

$$-\frac{4}{M}(2(z - z_0) + \ln\,(1 + \mathrm{e}^{-2(z-z_0)})) - \tfrac{1}{2}M\,h_1 = \frac{\mathrm{d}}{\mathrm{d}z}(h_0^{-1}h_1)$$

from which it follows that

$$h_1 \sim -8(2z - 1 - 2z_0)/M^2 \quad \text{as} \quad z \to +\infty,$$

so we deduce from (5.30) that

$$z_0 = x_0 - 1.$$

Returning to the original variables we finally obtain

$$\left.\begin{aligned} u \sim 4(t_c-t)^2/M(r^2+r_0^2 \exp(M/(t_c-t)-2)) \quad \text{as} \quad t \to t_c^- \\ \text{for} \quad r = O(\exp(M/2(t_c-t))) \end{aligned}\right\} \tag{5.32}$$

from (5.31), which may be written in the self-similar form

$$u \sim (t_c-t)^2 \exp\left(-\frac{M}{(t_c-t)}\right) f\left(r/\exp\left(\frac{M}{2(t_c-t)}\right)\right), \tag{5.33}$$

while

$$u \sim M/r^2 \ln^2 r \quad \text{as} \quad t \to t_c^- \quad \text{for} \quad (t_c-t)\ln r > \tfrac{1}{2}M, \tag{5.34}$$

which follows from (5.24).

Although the asymptotic behaviour given by (5.33) has similarities with (5.2), there are a number of crucial differences; among these is that the solution (5.33) extinguishes in finite time whereas (5.2) does not.

6. Discussion

This paper has concentrated on the radially symmetric equation (1.2). There is also interest (see, for example, Kamin & Rosenau 1981) in inhomogeneous nonlinear diffusion equations such as

$$\frac{\partial u}{\partial t} = x^l \frac{\partial}{\partial x}\left(x^k D(u) \frac{\partial u}{\partial x}\right), \tag{6.1}$$

and our results are also applicable to equations of this form; writing

$$r = x^{1-(k+l)/2}/(1-\tfrac{1}{2}(k+l))$$

transforms (6.1) to a radially symmetric equation with dimension

$$N = 2(1-l)/(2-(k+l)).$$

Applications such as these motivate the consideration of (1.2) with $N \neq 1, 2, 3$. The results given earlier are applicable to the case $N \geqslant 2$. In the range $0 < N < 2$ the results given in the Appendix indicate that finite mass solutions exist only for $n < 2/N$. When this condition is satisfied the large-time behaviour for solutions to (1.9) and (1.10) which conserve mass is again given by (1.6) and (1.7), but for $1 \leqslant n < 2/N$ we do not have uniqueness because other solutions exist which do not conserve mass.

To obtain a physically based criterion for selecting among the possible solutions to (1.9) and (1.10) in the ranges for which non-uniqueness occurs, i.e. (assuming $N > 0$) for $N < 2$, $1 \leqslant n < 2/N$ and $N = 2$, $n = 1$, we consider the following regularized initial-boundary value problem

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} = \frac{1}{r^{N-1}} \frac{\partial}{\partial r}\left(r^{N-1} u^{-n} \frac{\partial u}{\partial r}\right), \\ &\text{at} \quad r = 0, \quad r^{N-1} u^{-n}\, \partial u/\partial r = 0, \\ &\text{as} \quad r \to +\infty, \quad u \to \epsilon, \\ &\text{at} \quad t = 0, \quad u = I(r) + \epsilon, \end{aligned}\right\} \tag{6.2}$$

in the limit $\epsilon \to 0^+$. The same approach was adopted in King (1992*b*, *c*) for the cases $N = 1$ with $n \geqslant 2$, $N = 2$ with $n > 1$ and $N > 2$ with $n \geqslant 1$ for which there are no finite mass solutions. Thus in King (1992*b*, *c*) the inclusion of $\epsilon > 0$ ensures that a solution exists; here it plays the different role of selecting one of the multiplicity of solutions which exist for $\epsilon = 0$. A physical motivation for (6.2) is that in applications the diffusivity will not in practice become unbounded as the concentration goes to zero; defining $u^* = u - \epsilon$ we have

$$D(u^*) = (u^* + \epsilon)^{-n}$$

which is representative of the required behaviour. An alternative regularization of a similar type is provided by the model (1.2) and (1.3).

In the limit $\epsilon \to 0$ with $t = O(1)$ we have leading-order problem

$$\left.\begin{array}{l} \dfrac{\partial u_0}{\partial t} = \dfrac{1}{r^{N-1}}\dfrac{\partial}{\partial r}\left(r^{N-1}u_0^{-n}\dfrac{\partial u_0}{\partial r}\right), \\ \text{at} \quad r = 0, \qquad r^{N-1}u_0^{-n}\,\partial u_0/\partial r = 0, \\ \text{as} \quad r \to +\infty, \qquad u_0 \to 0, \\ \text{at} \quad t = 0, \qquad u_0 = I(r), \end{array}\right\} \tag{6.3}$$

so that (1.9) and (1.10) are reproduced. We now consider the various possibilities.

(*a*) $0 < N < 2, \quad 1 \leqslant n < 2/N$

(i)
$$u_0 \sim (nr^2/2(2-nN)\,t)^{-1/n} \quad \text{as} \quad r \to \infty, \tag{6.4}$$

(see (A 1)). The outer region is then given by

$$r = \epsilon^{-n/2}R, \quad u = \epsilon w,$$

so that

$$\left.\begin{array}{l} \dfrac{\partial w_0}{\partial t} = \dfrac{1}{R^{N-1}}\dfrac{\partial}{\partial R}\left(R^{N-1}\,w_0^{-n}\dfrac{\partial w_0}{\partial R}\right), \\ \text{as} \quad R \to 0^+, \qquad w_0 \to +\infty, \\ \text{as} \quad R \to +\infty, \qquad w_0 \to 1, \\ \text{as} \quad t = 0, R > 0, \qquad w_0 = 1. \end{array}\right\} \tag{6.5}$$

The solution to (6.5) is uniquely specified; evidence for this may be deduced by reformulating the problem by using the non-local transformation

$$v = -\int_R^\infty R^{N-1}(w_0 - 1)\,\mathrm{d}R + \frac{1}{N}R^n, \quad y = R^{2-N}.$$

It is easily shown that, for $n \neq 1$, $y(v,t)$ satisfies

$$\frac{\partial y}{\partial t} = \frac{(2-N)^{2-n}}{(n-1)}\frac{\partial}{\partial v}\left(y^{2(N-1)(n-1)/(2-N)}\left(\frac{\partial y}{\partial v}\right)^{n-1}\right), \tag{6.6}$$

$$\left.\begin{array}{l} \text{as} \quad v \to -\infty, \quad y \to 0, \\ \text{as} \quad v \to +\infty, \quad y \sim (Nv)^{(2-N)/N}, \\ \text{at} \quad t = 0, \qquad y = (Nv)^{(2-N)/N}H(v); \end{array}\right\} \tag{6.7}$$

for $n = 1$ equation (6.6) is replaced by

$$\frac{\partial y}{\partial t} = (2-N)\frac{\partial}{\partial v}\left(\ln\left(y^{2(N-1)/(2-N)}\frac{\partial y}{\partial v}\right)\right).$$

The conditions (6.7) follow since

$$\text{as} \quad y \to 0^+, \qquad v \to -\infty,$$
$$\text{at} \quad t = 0, y > 0, \qquad v = y^{N/(2-N)}/N.$$

The equation (6.6) belongs to class discussed in, for example, Atkinson & Bouillet (1979); we note that $\partial y/\partial v > 0$ holds for the required solution.

Unlike (6.5), the formulation (6.6) and (6.7) does not contain a singular boundary condition and, although the conditions (6.7) are not of the form for which equations of the class (6.6) are usually discussed, we believe that (6.6) and (6.7) has a unique solution which takes the self-similar form

$$y = t^{(2-N)/2}\, \Omega(v/t^{N/2}).$$

It follows that w_0 takes the form

$$w_0 = w_0(R/t^{\frac{1}{2}}),$$

and the solution to (6.5) satisfies

$$w_0 \sim (nR^2/2(2-nN)\,t)^{-1/n} \quad \text{as} \quad r \to 0^+$$

and therefore matches with (6.4).

We note that, while the issue of non-uniqueness then arises only for $0 < N < 2$ with $1 \leqslant n < 2/N$, the preceding analysis is in fact applicable to the whole of the range $N > 0$ with $0 < n < 2/N$.

(ii) (I) $$n \neq 1, \quad u_0 \sim A(t)\, r^{(N-2)/(n-1)} \quad \text{as} \quad r \to \infty, \tag{6.8}$$

(see (A 2)). This asymptotic form for u_0 implies the outer scalings

$$r = \epsilon^{(n-1)/(N-2)}\, R, \quad u = \epsilon w \tag{6.9}$$

giving the leading-order problem

$$\left.\begin{aligned} &\frac{\partial}{\partial R}\left(R^{N-1}\, w_0^{-n}\, \frac{\partial w_0}{\partial R}\right) = 0, \\ &\text{as} \quad R \to 0^+, \qquad w_0 \sim A(t)\, R^{(N-2)/(n-1)}, \\ &\text{as} \quad R \to +\infty, \quad w_0 \to 1. \end{aligned}\right\} \tag{6.10}$$

However, the problem (6.10) has no solution and this indicates that the outer problem (6.3) is to be solved subject to (6.4) rather than (6.8), thereby specifying u_0 uniquely. The solution to (6.2) which is selected in the limit $\epsilon \to 0^+$ is therefore that which satisfies (A 1), which in fact gives the maximal solution. We note, however, that non-maximal solutions can be selected in applications involving different boundary conditions; an example is given in King (1992*e*).

(ii) (II) $$n = 1, \quad u_0 \sim A(t) \exp\left(-b(t)\, r^{2-N}\right) \quad \text{as} \quad r \to \infty$$

(see (A 4)). The outer scalings would then be

$$r = (\ln(1/\epsilon))^{1/(2-N)}\, R, \quad u = \exp\left(-\ln(1/\epsilon)\, \phi\right),$$

with

$$\begin{aligned} &\frac{\partial}{\partial R}\left(R^{N-1}\, \frac{\partial \phi_0}{\partial R}\right) = 0, \\ &\text{as} \quad R \to 0^+, \quad \phi_0 \sim b(t)\, R^{2-N}, \\ &\text{as} \quad R \to \infty, \quad \phi_0 \to 1, \end{aligned}$$

which has no solution, implying that the maximal solution is also selected in the case $n = 1$.

(*b*) $N = 2, n = 1$

(i) $$u_0 \sim 2t/r^2 \ln^2 r \quad \text{as} \quad r \to \infty \tag{6.11}$$

(see (A 6)). In this case there are two further regions in the asymptotic structure. In the outer region we write

$$r = \epsilon^{-\frac{1}{2}} R, \quad u = \epsilon w$$

and recover (6.5) with $N = 2$, $n = 1$. The solution takes the form

$$w_0 = w_0(R/t^{\frac{1}{2}})$$

and satisfies $$w_0 \sim 2t/R^2 \ln^2 (1/R) \quad \text{as} \quad R \to 0^+. \tag{6.12}$$

In this case the similarity ordinary differential equation for w_0 can be integrated exactly; if we write

$$w_0 = t \exp(g(\zeta))/R^2,$$

where $\zeta = \ln(R/t^{\frac{1}{2}})$, then g satisfies

$$\mathrm{e}^g \left(1 - \frac{1}{2}\frac{\mathrm{d}g}{\mathrm{d}\zeta}\right) = \frac{\mathrm{d}^2 g}{\mathrm{d}\zeta^2},$$

$$\text{as} \quad \zeta \to -\infty, g \sim -2\ln(-\zeta),$$

$$\text{as} \quad \zeta \to +\infty, g = 2\zeta + o(1),$$

so that $$\frac{\mathrm{d}g}{\mathrm{d}\zeta} + 2\ln\left(1 - \frac{1}{2}\frac{\mathrm{d}g}{\mathrm{d}\zeta}\right) = -\tfrac{1}{2}\mathrm{e}^g. \tag{6.13}$$

Because of the logarithmic terms, expressions (6.11) and (6.12) do not match directly and a transition region is needed. The relevant variables here are

$$X = \ln r/\ln(1/\epsilon), \quad C = \ln^2(1/\epsilon)\, r^2 u$$

and at leading order

$$\left.\begin{aligned} &\frac{\partial C_0}{\partial t} = \frac{\partial}{\partial X}\left(C_0^{-1}\frac{\partial C_0}{\partial X}\right), \\ &\text{as} \quad X \to 0^+, \quad C_0 \sim 2t/X^2, \\ &\text{as} \quad X \to \tfrac{1}{2}^-, \quad C_0 \sim 2t/(\tfrac{1}{2} - X)^2, \\ &\text{at} \quad t = 0, \quad C_0 = 0, \end{aligned}\right\} \tag{6.14}$$

where we have matched with (6.11) and (6.12). The solution to (6.14) may be obtained exactly in the separable form

$$C_0 = 8\pi^2 t/\sin^2(2\pi X). \tag{6.15}$$

We note that these results hold only for $t < \frac{1}{2}M$ and that if this condition is met then

$$\int_0^\infty r u_0 \,\mathrm{d}r = M - 2t, \quad \int_0^\infty R(w_0 - 1)\,\mathrm{d}R = 2t,$$

showing how mass is transferred from $r = O(1)$ to $R = O(1)$. It is also worth noting that the asymptotic results just given cannot be matched directly into the late-stage behaviour for which the appropriate scalings are

$$t = \tfrac{1}{2}M + O(1), \quad r = O(\epsilon^{-\frac{1}{2}}), \quad u = O(\epsilon).$$

The intermediate asymptotic timescale describing the transition between the timescales $t < \frac{1}{2}M$ and $t > \frac{1}{2}M$ has scaling

$$t = \tfrac{1}{2}M + \hat{t}/\ln(1/\epsilon)$$

and it can be shown that for $\hat{t} = O(1)$ the leading-order behaviour in the limit $\epsilon \to 0$ takes the very unusual self-similar form

$$u \sim (\hat{t}^2 + \pi^2 M^2)\exp(-2q(\hat{t};\epsilon)\ln(1/\epsilon)) f(r/\exp(q(\hat{t};\epsilon)\ln(1/\epsilon)))/\ln^2(1/\epsilon)$$

with

$$q = \arctan(\pi M/(-\hat{t})) + o(1) \quad \text{as} \quad \epsilon \to 0$$

and
$$f(\eta) = 4/(M(1+\eta^2)).$$

(ii) $$u_0 \sim A(t)\, r^{-b(t)} \quad \text{as} \quad r \to \infty,$$

with $b(t) > 2$ (see (A 7)). The outer scalings would now be

$$r = \epsilon^{-1/b} R, \quad u = \epsilon w$$

so at leading order

$$\frac{\partial}{\partial R}\left(R\, w_0^{-1} \frac{\partial w_0}{\partial R}\right) = 0,$$

$$\text{as} \quad R \to 0^+, \qquad w_0 \sim A(t)\, R^{-b(t)},$$

$$\text{as} \quad R \to +\infty, \quad w_0 \to 1,$$

which has no solution. Hence the maximal solution, in this case satisfying (A 6), is also selected in the case $N = 2, n = 1$.

(c) $N > 2, 2/N < n < 1$

It is instructive to consider this case also even though there is no issue of selection, the solution to (6.2) with $\epsilon = 0$ being uniquely specified. The results (6.8)–(6.10) are valid but the crucial difference is that in this range (6.10) does possess a solution, namely

$$w_0 = (1 + (A(t))^{(1-n)} R^{-(N-2)})^{1/(1-n)}.$$

In this case there is also a third region with

$$R = \rho/\epsilon^{(nN-2)/(N-2)}, \quad w = 1 + \epsilon^{(nN-2)/2}\Phi$$

and at leading order

$$\frac{\partial \Phi_0}{\partial t} = \frac{1}{\rho^{N-1}}\frac{\partial}{\partial \rho}\left(\rho^{N-1}\frac{\partial \Phi_0}{\partial \rho}\right),$$

$$\text{as} \quad \rho \to 0^+, \quad \Phi_0 \sim \frac{1}{1-n}(A(t))^{(1-n)}\rho^{-(N-2)},$$

$$\text{as} \quad \rho \to +\infty, \Phi_0 \to 0,$$

$$\text{at} \quad t = 0, \qquad \Phi_0 = 0;$$

the mass being lost from $r = O(1)$ for $t < t_c$ appears in $\rho = O(1)$.

The following comments are intended to summarize the various cases of (1.9) and (1.10) for $N > 0$ with compactly supported initial data of finite mass (1.11).

1. $n < \min(1, 2/N)$. The solution is unique and its large-time behaviour takes the form (1.6).

2. $2/N > n \geqslant 1$, $N < 2$. The solution is not uniquely specified. The large-time behaviour of the maximal solution, which preserves mass, take the form (1.6).

3. $1 > n > 2/N, N > 2$. The solution is unique and extinguishes in finite time. The behaviour close to extinction can be determined from (4.1)–(4.3).

4. $n = 2/N, N > 2$. The solution is unique and the large-time behaviour is given by (2.32) and (2.34).

5. $n = 1$, $N = 2$. The solution is not uniquely specified and extinguishes in finite time. The maximal solution does not preserve mass and its behaviour close to extinction is given by (5.32) and (5.34).

6. $n \geqslant \max(1, 2/N)$, $N \neq 2$ and $n > 1$, $N = 2$. There are no finite mass solutions.

In cases (1)–(3) the asymptotic behaviour is given by a group-invariant solution to (1.9). Such results can therefore be used in conjunction with comparison theorems to obtain bounds on more general solutions. In cases (4) and (5) the similarity form describing the asymptotic behaviour does not exactly satisfy (1.9) and cannot therefore be directly used in this manner.

The very special status of case (5) may be indicated by noting that the other five ranges listed above intersect at the point $n = 1, N = 2$. Case (5) is also notable from the point of view of its symmetry group, it being invariant under an infinite dimension group which corresponds to conformal mappings (Nariboli 1979; King 1992*a*). In addition, it is possible to construct numerous exact non-group-invariant solutions to (1.1) in this special case (King 1992*d*).

We have concentrated in this paper on the radially symmetric equation (1.9). We expect that the large-time behaviour, or the behaviour close to extinction, of solutions to (1.1) will in general be radially symmetric, so that such results are more generally applicable. We also expect that the far-field behaviour of solutions to (1.1) (for compactly supported initial conditions at least) will be radially symmetric, so that our conclusions concerning the flux of material to infinity remain valid. In, for example, the finite-time extinction case discussed in §4 additional issues nevertheless arise in considering (1.1), such as the determination of the location of extinction in the case $n > 4/(N+2)$ in which the asymptotic profile contracts down onto a single point.

We conclude by briefly noting some of the consequences of our results for the model (1.2) and (1.3) in which m is an integer with $m \geqslant 2$. The high concentration behaviour of this model is governed by (1.1) and (1.4). In one and two dimensions the intermediate asymptotic behaviour (which describes the transition from high to intermediate peak concentrations) is given by (1.6) and (1.7) and applies for t large (but not too large). The decay of the peak concentration is therefore gradual and its time dependence can be characterized by (1.6). In three dimensions a much greater range of behaviour is possible. For $m = 2$ the intermediate asymptotic behaviour is again given by (1.6) and (1.7). For $m \geqslant 4$, however, the high concentration region will disappear abruptly at some finite time $t = t_c$, the time dependence in this case being characterized by (4.1)–(4.4). For $m = 4$ we have $\beta < 0$ so that the region in which $u/u_{\max} = O(1)$ continues to spread out as t approaches t_c. For $m = 5$ we have $\beta = 0$, while for $m \geqslant 5$ we have $\beta > 0$ implying that the high concentration region contracts down to a single point as it approaches extinction. Finally, for $m = 3$ the large t behaviour of the high concentration region takes the form (2.33) with $N = 3$, indicating a very rapid decay rate in the intermediate asymptotic régime.

Appendix. Far-field behaviour

This appendix is concerned with possible behaviours as $r \to \infty$ of solutions to (1.9) with $n > 0$, $N > 0$ for which the finite mass condition

$$\int_0^\infty r^{N-1} u \, \mathrm{d}r < \infty$$

is satisfied. There are two main possibilities, namely separable behaviour and quasi-steady behaviour. The nature of the solutions may be classified according to whether the flux to infinity,

$$J(t) = -\lim_{r\to\infty} \left(r^{N-1} u^{-n} \frac{\partial u}{\partial r} \right),$$

is zero (so that mass is conserved) or positive (so that mass is lost). Finite mass solutions are possible only for $n < \max(1, 2/N)$ for $N \neq 2$ and for $n \leqslant 1$ for $N = 2$ and we may distinguish the following cases.

(*a*) $n < \min(1, 2/N)$

Here we have a separable balance

$$u \sim (nr^2/2(2-nN)\,t)^{-1/n} \quad \text{as} \quad r \to \infty, \tag{A 1}$$

so that $J(t) = 0$ and mass is conserved.

(*b*) $N < 2, 1 < n < 2/N$

There are now two possibilities. Expression (A 1) can again be valid and such behaviour gives maximal (mass-preserving) solutions. A quasi-steady balance

$$u \sim A(t)\, r^{(N-2)/(n-1)} \quad \text{as} \quad r \to \infty \tag{A 2}$$

is also possible with

$$J(t) = -\frac{N-2}{n-1} A(t). \tag{A 3}$$

In this range the solution is not uniquely specified unless $J(t)$ is prescribed; (A 1) corresponds to the case $J(t) = 0$.

(*c*) $N < 2, n = 1$

Expressions (A 1) and the quasi-steady balance

$$u \sim A(t) \exp(-b(t)\, r^{2-N}) \quad \text{as} \quad r \to \infty, \tag{A 4}$$

with

$$J(t) = -(N-2)\, b(t),$$

are both possible. As with case (*b*), the flux $J(t)$ must be prescribed if the problem is to be completely specified.

(*d*) $N > 2,\ 2/N < n < 1$

Expression (A 2) describes the behaviour in this case since (A 1) is not admissible. The flux $J(t)$ is positive and is determined as part of the solution.

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(*e*) $N>2$, $n=2/N$

As in case (*a*), we are restricted to a separable balance; here we have

$$u \sim (r^2 \ln r/(N-2)\,t)^{-N/2} \quad \text{as} \quad r \to \infty, \tag{A 5}$$

for which $J(t)=0$.

(*f*) $N=2$, $n=1$

As noted earlier, the possibilities here are separable with

$$u \sim (r^2 \ln^2 r/2t)^{-1} \quad \text{as} \quad r \to \infty, \tag{A 6}$$

for which $J(t)=2$, and quasi-steady with

$$u \sim A(t)\, r^{-b(t)} \quad \text{as} \quad r \to \infty, \tag{A 7}$$

with $b(t)>2$ and $J(t)=b(t)$. The flux $J(t)$ must be specified for the solution to be uniquely determined; here we need $J(t) \geqslant 2$ whereas in cases (*b*) and (*c*) we require $J(t) \geqslant 0$.

The cases listed above do not exhaust the possibilities. For example, if

$$I(r) \sim Ar^{-b} \quad \text{as} \quad r \to \infty, \tag{A 8}$$

with $b<2/n$ then at leading order we have steady state behaviour with

$$u(r,t) \sim Ar^{-b} \quad \text{as} \quad r \to \infty.$$

However, the possibilities listed above are those appropriate when (for example) $I(r)$ has compact support.

In the remaining range, namely $n \geqslant \max(1, 2/N)$ for $N \neq 2$ and $n>1$ for $N=2$, consistent far-field balances include

$$u \sim (nr^2/(nA^n - 2(nN-2)\,t))^{-1/n} \quad \text{as} \quad r \to \infty \tag{A 9}$$

and

$$u \sim Ar^{-b} \quad \text{as} \quad r \to \infty \quad \text{for} \quad b<2/n,$$

both of which arise from initial conditions satisfying (A 8) with $b \leqslant 2/n$ (when $n>2/N$, the form (A 9) implies finite time extinction). All such possibilities require infinite mass initial conditions and we infer that, as already noted, no finite mass solutions exist in this range.

References

Aronson, D. G. 1986 The porous media equation. In *Nonlinear diffusion problems* (ed. A. Fasano & M. Primicerio). Berlin: Springer-Verlag.

Aronson, D. G., Graveleau, J. & Souganidis, J. 1992 A self-similar solution to the focusing problem for the porous medium equation. (Preprint.)

Atkinson, C. & Bouillet, J. E. 1979 Some qualitative properties of solutions of a generalised diffusion equation. *Math. Proc. Camb. phil. Soc.* **86**, 495–510.

Benilan, P. & Crandall, M. G. 1981 The continuous dependence on ϕ of solutions of $u_t - \Delta\phi(u) = 0$. *Indiana Univ. Math. J.* **30**, 161–177.

Bernis, F., Hulshof, J. & Vazquez, J. L. 1992 A very singular solution for the dual porous medium equation and the asymptotic behaviour of general solutions. (Preprint.)

Berryman, J. G. & Holland, C J. 1978 Nonlinear diffusion problem arising in plasma physics. *Phys. Rev. Lett.* **40**, 1720–1722.

Berryman, J. G. & Holland, C. J. 1982 Asymptotic behaviour of the nonlinear diffusion equation $n_t = (n^{-1} n_x)_x$. *J. math. Phys.* **23**, 983–987.

Brezis, H. & Friedman, A. 1983 Nonlinear parabolic equations involving measures as initial conditions. *J. Math. pures appl.* **62**, 73–97.

de Gennes, P. G. 1984 Spreading laws for microscopic droplets. *C.r. Acad. Sci., Paris II* **298**, 475–478.

Esteban, J. R., Rodriguez, A. & Vazquez, J. L. 1988 A nonlinear heat equation with singular diffusivity. *Comm. PDE* **13**, 985–1039.

Friedman, A. & Kamin, S. 1980 The asymptotic behaviour of gas in an n-dimensional porous medium. *Trans. Am. math. Soc.* **262**, 551–563.

Galaktionov, V. A. & Posashkov, S. A. 1986 On the nonlinear 'fast diffusion' equation in R^N. *Sov. Math. Dokl.* **33**, 412–415.

Herrero, M. A. 1989 A limit case in nonlinear diffusion. *Nonlin. Analysis* **13**, 611–628.

Kamin, S. & Rosenau, P. 1981 Propagation of thermal waves in an inhomogeneous medium. *Communs pure appl. math.* **34**, 831–852.

Kath, W. L. & Cohen, D. S. 1982 Waiting-time behaviour in a nonlinear diffusion equation. *Stud. appl. Math.* **67**, 79–105.

King, J. R. 1988 Extremely high concentration dopant diffusion in silicon. *IMA J. appl. Math.* **40**, 163–181.

King, J. R. 1992*a* Local transformations between some nonlinear diffusion equations. *J. Aust. math. Soc.* B **33**, 321–349.

King, J. R. 1992*b* 'Instantaneous source' solutions to a singular nonlinear diffusion equation. *J. Engng Math.* (In the press.)

King, J. R. 1992*c* Multidimensional singular diffusion. (Submitted.)

King, J. R. 1992*d* Exact polynomial solutions to some nonlinear diffusion equations. *Physica* D. (In the press.)

King, J. R. 1992*e* Asymptotic results for nonlinear outdiffusion. (Submitted.)

Landau, L. D. & Lifschitz, E. M. 1959 *Fluid Mechanics*, 1st edn. Oxford: Pergamon.

Lonngren, K. E. & Hirose, A. 1976 Expansion of an electron cloud. *Phys. Lett.* A **59**, 285–286.

Lopez, J., Miller, C. A. & Ruckenstein, E. 1976 Spreading kinetics of liquid drops on solids. *J. Coll. Int. Sci.* **56**, 460–468.

Nariboli, G. A. 1970 Self-similar solutions of some nonlinear equations. *Appl. Sci. Res.* **22**, 449–461.

Rodriguez, A. & Vazquez, J. L. 1990 A well posed problem in singular Fickian diffusion. *Arch. ration. Mech. Analysis* **110**, 141–163.

Ryssel, H., Haberger, J., Hoffman, K., Princke, G., Dumcke, R. & Sachs, A. 1980 Simulation of doping processes. *IEEE Trans. Electron Dev.* **ED-27**, 1484–1492.

Starov, V. M. 1983 Spreading of droplets of nonvolatile liquids over a flat solid surface. *Coll. J. USSR* **45**, 1009–1014.

Vazquez, J. L. 1992 Nonexistence of solutions for nonlinear heat equations of fast-diffusion type. (Preprint.)

Zel'dovich, Ia. B. & Barenblatt, G. I. 1958 The asymptotic properties of self-modelling solutions of the non-stationary gas filtration equations. *Sov. Phys. Dokl.* **3**, 44–47.

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